Lecture 20

Inverses

a special case of a concept of great importance in Maths

We now turn to deal with the case where we can <u>divide</u> one matrix by another.

<u>Definition</u> Let A be a <u>square matrix</u>

We say that A is <u>invertible</u> if there is a square matrix B such that

$$AB = I = BA$$

identity matrix

We call B <u>an inverse of A</u>

<u>Example</u> The matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ is invertible. Take $B = -\frac{1}{5}\begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix}$

$$AB = -\frac{1}{5}\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix}$$

$$= -\frac{1}{5}\begin{pmatrix} 1-6 & -2+2 \\ 3-3 & -6+1 \end{pmatrix}$$

$$= -\frac{1}{5}\begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Check that $BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ also.

<u>Example</u> You can check that $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is <u>not</u> invertible.

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$! \begin{cases} a+c=1, & b+d=0, \\ a+c=0, & b+d=1 \end{cases}$$

<u>Not possible</u>

<u>Lemma</u> Let A be an invertible matrix. Suppose that

$$AB = I = BA$$

$$AC = I = CA$$

Then $C = B$.

<u>Proof</u> $\quad AB = I$

$$\therefore \quad C(AB) = CI = C$$

$$\begin{aligned} C(AB) &= ((CA)B) \\ &= IB \\ &= B \end{aligned}$$

$\therefore B = C$ ■

Because of the above lemma, if a matrix A has an inverse, it has a unique inverse. We denote this unique inverse (whenever it exists) by A^{-1}.

<u>Example</u> Let $A\underline{x} = \underline{b}$ where A is invertible. Then

$$A^{-1}(A\underline{x}) = A^{-1}\underline{b}$$

$$\begin{aligned} A^{-1}(A\underline{x}) &= (A^{-1}A)\underline{x} \\ &= I\underline{x} \\ &= \underline{x} \end{aligned}$$

$\therefore \boxed{\underline{x} = A^{-1}\underline{b}.}$

Observe that the $\underline{b}$ can vary ~~and~~ but we can still easily solve the new equations using A^{-1} (rather than going back to square one)

We would like to answer the following two questions:

(1) How do we decide whether a matrix is invertible or not?

(2) If a matrix is invertible how do we compute its inverse.

We shall answer both of these questions concentrating on 2×2 & 3×3 matrices (though everything generalizes).

The determinant

This will help us answer question 1.
(& 2, as well, actually).

<u>Definition</u> Let A be a <u>square matrix</u>.

We define the <u>determinant</u> of A, written $\det(A)$

or $\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{n1} & \dots & & a_{nn} \end{vmatrix}$, as follows.

$$\det(a) = a$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \underline{\underline{a}}\begin{vmatrix} e & f \\ h & i \end{vmatrix} - \underline{\underline{b}}\begin{vmatrix} d & f \\ g & i \end{vmatrix} + \underline{\underline{c}}\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

¡ Pay attention to <u>signs</u>!

<u>Exaple</u>

(1) $$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 2 \times 5 - 3 \times 4$$
$$= 10 - 12 = \underline{\underline{-2}}$$

(2) $$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix}$$
$$+ 1 \begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix}$$

$$= 1 - 2(3) + (-2)$$
$$= 1 - 6 - 2 = \underline{\underline{-7}}$$

The key property we shall need in this section is the following. I shall not prove it in generality but the proof in the 2×2 case can be found on pp 249-250 of my book.

Theorem Let A and B be $n \times n$ matrices. Then $\det(AB) = \det(A)\det(B)$.

This leads immediately to the following.

Lemma If A is invertible then $\det(A) \neq 0$.

Proof. We have that $AA^{-1} = I$.

Thus $\det(AA^{-1}) = \det(I) = 1$

But $\det(AA^{-1}) = \det(A)\det(A^{-1})$.

It follows that $\det(A) \neq 0$. ∎