Lecture 21

It turns out that this is, in fact, an exact criterion:

$$\boxed{A \text{ is invertible} \iff \det(A) \neq 0}$$

I shall prove this in the 2×2 and 3×3 cases by explicitly computing A^{-1}.

2×2 Case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We construct a matrix called the adjugate of A, written $\mathrm{adj}(A)$, with the property that

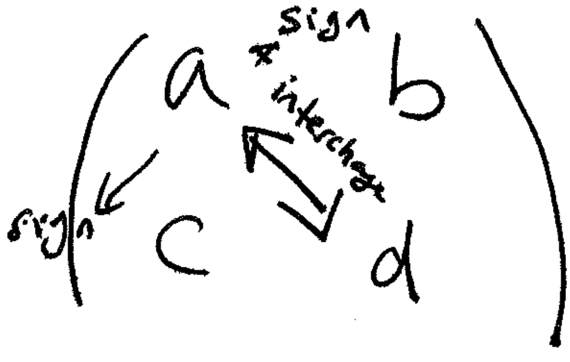
$$A\,\mathrm{adj}(A) = \det(A)\,I = \mathrm{adj}(A)\,A.$$

<u>Input</u> $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

<u>Procedure</u> • For each entry, cross out its row and column and write down the number you see

$$\begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

• Add signs $(-1)^{i+j}$ where i is row number and j is column number



$$\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

• "TAKE THE TRANSPOSE"

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \text{adj}(A).$$

Check

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} =$$

$$\begin{pmatrix} ad-bc & -ba+ba \\ cd-dc & -bc+ad \end{pmatrix}$$

$$= \det(A) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It follows that if $\det(A) \neq 0$ then

$$\boxed{A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)}$$

3 x 3 case

This is identical to the 2 x 2 case except that in the first step, for each entry, cross out its row and column and write down the determinant of the matrix you see. The remaining steps are the same.

$$= \begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ -1 & 1 & 2 \end{pmatrix}^{-1} = \frac{1}{-5} \begin{pmatrix} -1 & -1 & 2 \\ -5 & 5 & 5 \\ 2 & -3 & -4 \end{pmatrix}$$

In general, if $\det(A) \neq 0$ then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

Example We compute the adjugate of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

- $$\begin{pmatrix} \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 5 & 2 \\ 1 & 5 & 3 \\ 2 & -5 & -4 \end{pmatrix}$$

- Now multiply entry in row i and column j by $(-1)^{i+j}$

$$\begin{pmatrix} -1 & -5 & 2 \\ -1 & 5 & -3 \\ 2 & 5 & -4 \end{pmatrix}$$

- NOW TAKE THE TRANSPOSE

$$\begin{pmatrix} -1 & -1 & 2 \\ -5 & 5 & 5 \\ 2 & 3 & -4 \end{pmatrix} = \operatorname{adj}(A)$$

<u>Check</u>

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} -1 & -1 & 2 \\ -5 & 5 & 5 \\ 2 & -3 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} -1-10+6 & -1+10-9 & 2+10-12 \\ -2+2 & -2-3 & 2+10-12 \\ 1-5+4 & 1+5-6 & -2+5-8 \end{pmatrix}$$

The characteristic polynomial

This section is just to give a flavor (maybe, a smell) of Matrix theory.

Definition. Let A be a square matrix.

$$\chi_A(\lambda) = \det(A - \lambda I)$$

is a polynomial called the characteristic polynomial of A. The roots of $\chi_A(\lambda)$ are called the eigenvalues of A.

These contain important information about A useful in many applications.

<u>Example</u> Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\chi_A(\lambda) = \det(A - \lambda I)$$

$$= \det \begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & (1-\lambda) \end{pmatrix}$$

$$= \underline{\underline{(1-\lambda)^3}}$$

Eigenvals = 1 (three times).