Lecture 22

Theorem (Cayley-Hamilton).
Every square matrix A is a root of its characteristic polynomial.

We shall not prove the above theorem, but we shall illustrate it by means of an example.

Example Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$.

Its characteristic polynomial is

$$\chi_A(\lambda) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda - 1$$

To understand what $\chi_A(A)$ means we need to remember that

$$x^2-4x-1 = x^2-4x^1-x^0$$

and define $A^0 = I$ (identity matrix same size as A).

So $\chi_A(A) = A^2 - 4A - I$

$$\begin{pmatrix}5&8\\8&13\end{pmatrix} - \begin{pmatrix}4&8\\8&12\end{pmatrix} - \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$= \begin{pmatrix}1&2\\2&3\end{pmatrix}^2 - 4\begin{pmatrix}1&2\\2&3\end{pmatrix} - \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$= \begin{pmatrix}0&0\\0&0\end{pmatrix}$$ ✓

So, A is a root of $\chi_A(A)$. ∎

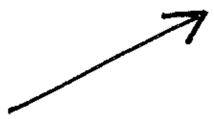
$\boxed{\text{END OF SECTION 3}}$

Section 4

Vectors

We view vectors first geometrically and then algebraically.

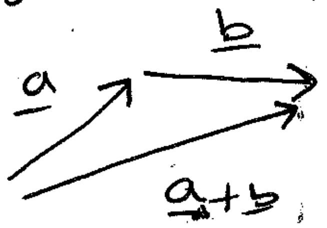
Definition A vector is a directed line segment which can be moved parallel to itself.



Vectors are denoted by bold letters $\underline{a}, \underline{b}, \underline{c}, \ldots$

If P and Q are two points in space then the directed line segment from P to Q is denoted by $\overrightarrow{PQ}$. This determines a vector. If $P=Q$ the directed line segment $\overrightarrow{PP}$ is just a point. This determines the zero vector $\underline{0}$.

If $\underline{a}$ is a vector then $-\underline{a}$ is the vector pointing in the opposite direction. If $\underline{a}$ and $\underline{b}$ are vectors we may add them by translating $\underline{a}$ so it points at the start of $\underline{b}$.



Properties of vector addition

1. $\underline{a} + (\underline{b} + \underline{c}) = (\underline{a} + \underline{b}) + \underline{c}$.
2. $\underline{0} + \underline{a} = \underline{a} = \underline{a} + \underline{0}$.
3. $\underline{a} + (-\underline{a}) = \underline{0} = (-\underline{a}) + \underline{a}$.
4. $\underline{a} + \underline{b} = \underline{b} + \underline{a}$.

subtraction of vectors is defined by $\underline{a} - \underline{b} \equiv \underline{a} + (-\underline{b})$.