Lecture 23

Properties of vector addition

1. $\underline{a} + (\underline{b} + \underline{c}) = (\underline{a} + \underline{b}) + \underline{c}$.
2. $\underline{0} + \underline{a} = \underline{a} = \underline{a} + \underline{0}$.
3. $\underline{a} + (-\underline{a}) = \underline{0} = (-\underline{a}) + \underline{a}$.
4. $\underline{a} + \underline{b} = \underline{b} + \underline{a}$.

Subtraction of vectors is defined by $\underline{a} - \underline{b} \equiv \underline{a} + (-\underline{b})$.

Let $\underline{a}$ be a vector. Define $\| \underline{a} \|$ the length of the vector. $\|\underline{a}\| \geq 0$, and $\|\underline{a}\| = 0 \Leftrightarrow \underline{a} = \underline{0}$.

Can multiply a vector $\underline{a}$ by a scalar $\lambda \in \mathbb{R}$, to get $\lambda \underline{a}$. If $\lambda > 0$ then $\lambda \underline{a}$ has the same direction as $\underline{a}$ and is parallel. If $\lambda < 0$ then $\lambda \underline{a} = -(-\lambda)\underline{a}$.

If $\underline{a} \neq \underline{0}$ then $\hat{\underline{a}} = \frac{1}{\|\underline{a}\|} \underline{a}$ is a unit vector (a vector of length 1) pointing in the same direction as $\underline{a}$ and parallel.

Properties of scalar multiplication

1. $0\underline{a} = \underline{0}$. 2. $1\underline{a} = \underline{a}$. 3. $(-1)\underline{a} = -\underline{a}$.

4. $(\lambda + \mu)\underline{a} = \lambda\underline{a} + \mu\underline{a}$. 5. $\lambda(\underline{a} + \underline{b}) = \lambda\underline{a} + \lambda\underline{b}$.

6. $\lambda(\mu\underline{a}) = (\lambda\mu)\underline{a}$.

Operations

If $\underline{a}, \underline{b} \neq \underline{0}$ denote the angle between them by θ

$$0 \leq \theta \leq \pi$$

Define $\underline{a} \cdot \underline{b} = \|\underline{a}\| \|\underline{b}\| \cos\theta$

If $\underline{a}$ or $\underline{b}$ is $\underline{0}$ then $\underline{a} \cdot \underline{b} = 0$.

Called the inner product.

Properties of the inner product

1. $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$.
2. $\underline{a} \cdot \underline{a} = \|\underline{a}\|^2$.
3. $\lambda(\underline{a} \cdot \underline{b}) = (\lambda \underline{a}) \cdot \underline{b} = \underline{a} \cdot (\lambda \underline{b})$ where $\lambda \in \mathbb{R}$
4. $\boxed{\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}}$ See book for proof

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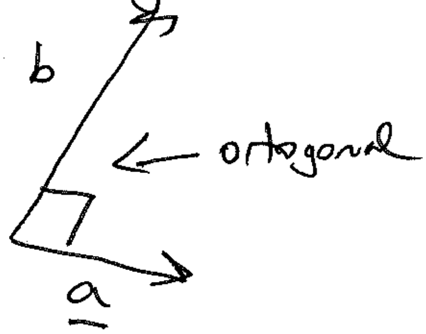
If $\underline{a}, \underline{b} \neq \underline{0}$ and $\underline{a} \cdot \underline{b} = 0$ we say that $\underline{a}$ and $\underline{b}$ are ortogonal.

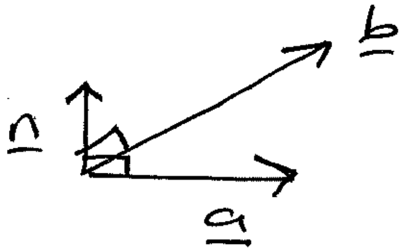


Let $\underline{a}$ and $\underline{b}$ be non-zero vectors.

area = $\|\underline{b}\| \sin\theta \, \|\underline{a}\|$

$$\underline{a} \times \underline{b} = \underbrace{\|\underline{a}\| \, \|\underline{b}\| \sin\theta}_{\text{scalar}} \; \underbrace{\underline{n}}_{\text{vector}}$$

Where $\underline{n}$ is a unit vector orthogonal to both $\underline{a}$ and $\underline{b}$ where $\underline{n}$ is chosen as below



If either $\underline{a}$ or $\underline{b}$ is $\underline{0}$ we define $\underline{a} \times \underline{b} = \underline{0}$.

We call $\underline{a} \times \underline{b}$ the vector product of $\underline{a}$ and $\underline{b}$.

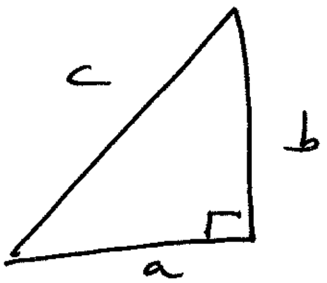
Properties of the vector product

1. $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$.
2. $\lambda(\underline{a} \times \underline{b}) = (\lambda \underline{a}) \times \underline{b} = \underline{a} \times (\lambda \underline{b})$ where $\lambda \in \mathbb{R}$.
3. $\boxed{\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}}$. proved in book

NB Vector product is not associative.

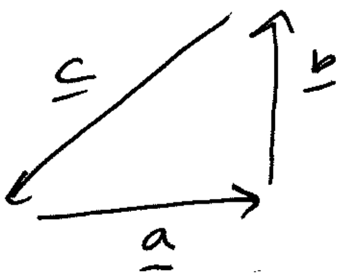
Application

We prove Pythagoras' theorem.



$$a^2 + b^2 = c^2$$

Choose vectors



$$\underline{a} + \underline{b} + \underline{c} = \underline{0}$$

$$\underline{a} + \underline{b} = -\underline{c}$$

where $\|\underline{a}\| = a$, $\|\underline{b}\| = b$

$\|\underline{c}\| = c$.

$$(\underline{a} + \underline{b}) \cdot (\underline{a} + \underline{b}) = \underline{c} \cdot \underline{c} = \|\underline{c}\|^2$$

$$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} = 0$$

$$\underline{a} \cdot \underline{a} + \boxed{\underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{a}} + \underline{b} \cdot \underline{b} = \|\underline{c}\|^2$$

$$\|\underline{a}\|^2 + \|\underline{b}\|^2 = \|\underline{c}\|^2$$

i.e. $a^2 + b^2 = c^2$