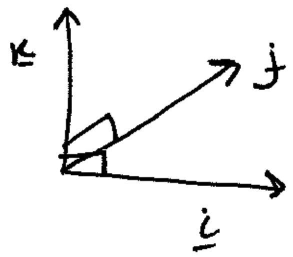
Lecture 24

Vectors algebraically

Choose three unit vectors $\underline{i}, \underline{j}, \underline{k}$ where $\underline{k} = \underline{i} \times \underline{j}$.



We introduce co-ordinates

Every vector $\underline{a}$ can now be written

$$\underline{a} = a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}$$

where a_1, a_2, a_3 are called the <u>components</u> of $\underline{a}$.

We can now obtain co-ordinate forms of the inner and vector product.

<u>Inner products</u> Let $\underline{a} = a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}$ and $\underline{b} = b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k}$. Then

$$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 .$$

<u>Proof</u>

$\cdot$	$\underline{i}$	$\underline{j}$	$\underline{k}$
$\underline{i}$	1	0	0
$\underline{j}$	0	1	0
$\underline{k}$	0	0	1

$$\underline{a} \cdot \underline{b} = (a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}) \cdot \underline{b}$$

$$= (a_1 \underline{i}) \cdot \underline{b} + (a_2 \underline{j}) \cdot \underline{b} + (a_3 \underline{k}) \cdot \underline{b}$$

$$= a_1 (\underline{i} \cdot \underline{b}) + a_2 (\underline{j} \cdot \underline{b}) + a_3 (\underline{k} \cdot \underline{b})$$

$$\underline{i} \cdot \underline{b} = \underline{i} \cdot (b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k})$$

$$= b_1$$

$$\underline{j} \cdot \underline{b} = \underline{j} \cdot (b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k})$$

$$= b_2$$

$$\underline{k} \cdot \underline{b} = b_3$$

It follows that $\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$ ∎

$$\underline{a} \cdot \underline{a} = a_1^2 + a_2^2 + a_3^2$$

$$\therefore \|\underline{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

<u>Example</u> Let $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$

and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$. $(\underline{a}, \underline{b} \neq \underline{0})$

Then $\underline{a} \cdot \underline{b} = \|\underline{a}\| \, \|\underline{b}\| \cos\theta$.

So, $$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{\|\underline{a}\| \; \|\underline{b}\|}$$

Thus

$$\boxed{\cos\theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}}$$

We can therefore easily compute the angle between two vectors written in co-ordinate form.

Vector products

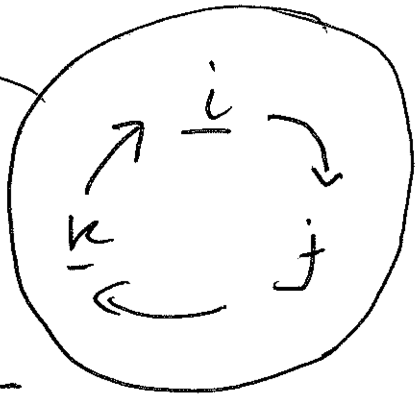
Let $\underline{a} = a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}$ and

$\underline{b} = b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k}$.

Then

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \underline{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \underline{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \underline{k}$$



Proof

$\times$	$\underline{i}$	$\underline{j}$	$\underline{k}$
$\underline{i}$	$\underline{0}$	$\underline{k}$	
$\underline{j}$		$\underline{0}$	$\underline{i}$
$\underline{k}$	$\underline{j}$		$\underline{0}$

Now use $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$ to fill in the rest.

$\times$	$\underline{i}$	$\underline{j}$	$\underline{k}$
$\underline{i}$	$\underline{0}$	$\underline{k}$	$-\underline{j}$
$\underline{j}$	$-\underline{k}$	$\underline{0}$	$\underline{i}$
$\underline{k}$	$\underline{j}$	$-\underline{i}$	$\underline{0}$

$$\underline{a}\times\underline{b} = \underline{a}\times(b_1\underline{i} + b_2\underline{j} + b_3\underline{k})$$

$$= b_1(\underline{a}\times\underline{i}) + b_2(\underline{a}\times\underline{j}) + b_3(\underline{a}\times\underline{k})$$

So we need to calculate $\underline{a}\times\underline{i}$, $\underline{a}\times\underline{j}$, and $\underline{a}\times\underline{k}$. I shall just do $\underline{a}\times\underline{i}$ here:

$$\underline{a}\times\underline{i} = (a_1\underline{i} + a_2\underline{j} + a_3\underline{k})\times\underline{i}$$

$$= a_1(\underline{i}\times\underline{i}) + a_2(\underline{j}\times\underline{i}) + a_3(\underline{k}\times\underline{i})$$

$$= -a_2\underline{k} + a_3\underline{j} \quad \blacksquare$$

Geometric meaning of determinants

Define $[\underline{a}, \underline{b}, \underline{c}] = \underline{a} \cdot (\underline{b} \times \underline{c})$.

Called the scalar ~~product~~ triple product.

Using our results for $\cdot$ and $\times$ we can easily show that

$$[\underline{a}, \underline{b}, \underline{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

[Let me mention a couple of important further properties of determinants

- $\det(A^T) = \det(A)$.
- If B is obtained from A by interchanging two columns then $\det(B) = -\det(A)$.]