Lecture 4

In the last lecture, we introduced the Boolean operations $A \cap B$, $A \cup B$, $A \setminus B$.

> Whenever you meet new operations, always ask what properties those operations have.

Please see the handout (overleaf) for a list of the important properties the Boolean operations satisfy. To understand them, you need to understand the following terms

- Binary operation
- Associative / associativity
- Commutative / Commutativity
- Distributive / Distributivity
- Idempotent / idempotence

defined in handout

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Properties of Boolean operations

This is a version of Section 3.2.

Key definitions

- Let X be a non-empty set. A *binary operation* $*$ on X associates with every ordered pair (x, y) of elements of X a single element $x * y$ of X.
- The binary operation $*$ is said to be *commutative* if $x * y = y * x$ for all $x, y \in X$.
- The binary operation $*$ is said to be *associative* if $(x * y) * z = x * (y * z)$ for all $x, y, z \in X$.
- An *identity* for $*$ is an element e such that $e * x = x = x * e$ for all $x \in X$.
- A *zero* for $*$ is an element z such that $z * x = z = x * z$ for all $x \in X$.
- The binary operation $*$ is said to be *idempotent* if $x * x = x$ for all $x \in X$.
- If $\bullet$ is another binary operation on the set X we say that $*$ *distributes over* $\bullet$ if $x * (y \bullet z) = (x * y) \bullet (x * z)$.

Properties

Let A, B and C be any sets.

(1) $A \cap (B \cap C) = (A \cap B) \cap C$. Intersection is associative.
(2) $A \cap B = B \cap A$. Intersection is commutative.
(3) $A \cap \emptyset = \emptyset = \emptyset \cap A$. The empty set is the zero for intersection.
(4) $A \cup (B \cup C) = (A \cup B) \cup C$. Union is associative.
(5) $A \cup B = B \cup A$. Union is commutative.
(6) $A \cup \emptyset = A = \emptyset \cup A$. The empty set is the identity for union.
(7) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Intersection distributes over union.
(8) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. Union distributes over intersection.
(9) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$. De Morgan's law part one.
(10) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$. De Morgan's law part two.
(11) $A \cap A = A$. Intersection is idempotent.
(12) $A \cup A = A$. Union is idempotent.

How do we ~~verify~~ Check these properties?

- Some can be checked by using the result that $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$

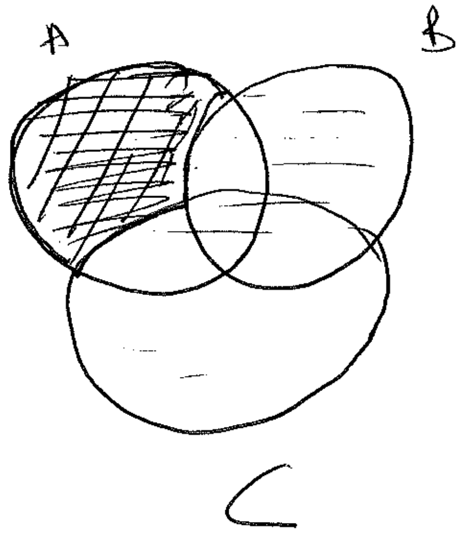
<u>Example</u> Property (3). $A \cap \emptyset = \emptyset = \emptyset \cap A$.

Clearly, $\emptyset \subseteq A \cap \emptyset$ since $\emptyset$ is a subset of any set. Let $a \in A \cap \emptyset$ then $a \in A$ and $a \in \emptyset$. But $a \in \emptyset$ is impossible. Thus $A \cap \emptyset$ is empty. It follows that $\emptyset = A \cap \emptyset$.

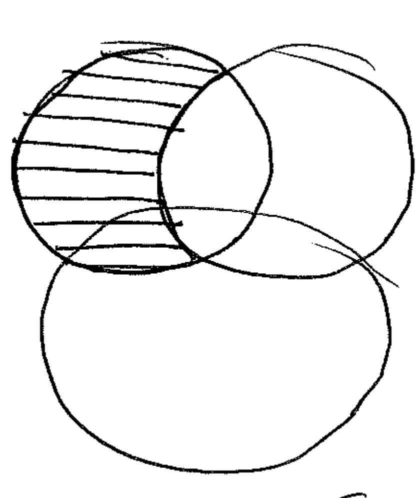
<u>Example</u> Property (2) and (5). follow from the fact that 'p and q' means the same (in Maths) as 'q and p'. Similarly, for 'or'.

To prove other properties, we need to know more logic. We shall content ourselves here with <u>illustrating</u> properties using Venn diagrams.

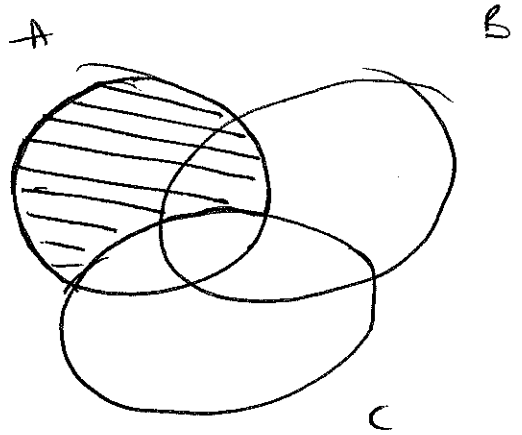
<u>Example</u> We illustrate $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

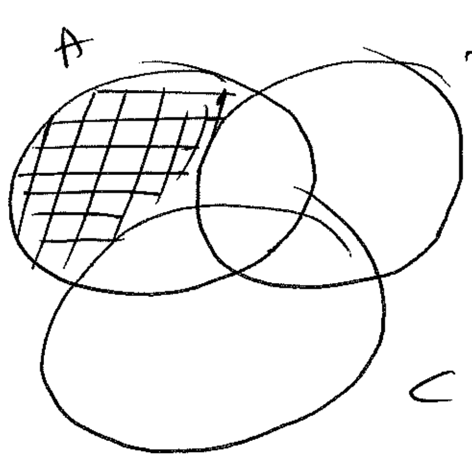


$A \setminus (B \cup C)$



$A \setminus B$





$(A \setminus B) \cap (A \setminus C)$

Important consequence of these properties

Because of associativity, we can write multiple unions and intersections without brackets

$$\left.\begin{array}{l} A_1 \cup A_2 \cup \dots \cup A_n \\ A_1 \cap A_2 \cap \dots \cap A_n \end{array}\right\} \text{unambiguous}$$

We would usually write

$\bigcup_{i=1}^{n} A_i$ instead of $A_1 \cup \dots \cup A_n$

$\bigcap_{i=1}^{n} A_i$ instead of $A_1 \cap \dots \cap A_n$.

Counting results

(Section 3.9)

We begin by listing the results and giving some examples of them we shall show how to prove them.

Theorem 1 Let X be any (finite) set. Then $|P(X)| = 2^{|X|}$.

Example $X = \{a,b,c\}$

$$P(\{a,b,c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$$

Then $|P(\{a,b,c\})| = 8 = 2^3 = 2^{|X|}$

as the above theorem predicts.

<u>Definition</u> Let X be a set with n elements. A subset $Y \subseteq X$ with k-elements $(0 \le k \le n)$ is called a <u>k-subset</u>

<u>Example</u> Let $X = \{a, b, c, d\}$

2-subsets.

We find all the

$\{a,b\}$, $\{a,c\}$, $\{a,d\}$

$\{b,c\}$, $\{b,d\}$,

$\{c,d\}$

6 ~~subsets~~ 2-subsets

This might seem an odd way of defining $n!$ but it will be useful later

<u>Definition</u> Let $n \geq 0$ be a natural no.

Define $0! = 1$ and if $n \geq 1$ define

$n! = n(n-1)!$, Called n <u>factorial</u>

<u>Example</u> $4! = 4 \cdot 3!$

$= 4 \cdot 3 \cdot 2!$

$= \cancel{4 \cdot 3 \cdot 2 \cdot 0!}\ 4 \cdot 3 \cdot 2 \cdot 1!$

$= 4 \cdot 3 \cdot 2 \cdot 1 \cdot \underline{0!} = 4 \cdot 3 \cdot 2 \cdot 1 = \underline{\underline{24}}$