Lecture 5

We were looking last time at k-subsets.

<u>Example</u> Let $X = \{a, b, c, d\}$. The 2-subsets are $\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}$ & there are 6 2-subsets of a 4-element set.

<u>Definition</u> Let $n \geq 0$ be a natural number.

$0! = 1$ and if $n \geq 1$ — seems odd, but is needed

$$n! = n(n-1)!.$$

$n!$ is called <u>n factorial</u>.

<u>Example</u>

$$\begin{aligned} 4! &= 4 \cdot 3! \\ &= 4 \cdot 3 \cdot 2! \\ &= 4 \cdot 3 \cdot 2 \cdot 1! \\ &= 4 \cdot 3 \cdot 2 \cdot 1 \cdot 0! \\ &= 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1 \\ &= 4 \cdot 3 \cdot 2 \cdot 1 = \underline{24} \end{aligned}$$

This is an example of defining a function by <u>recursion</u>

Theorem 2 Let X be a set with $|X| = n$. Then the number of k-subsets is

$$^nC_k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Read 'n choose k'

These numbers are called binomial coefficients.

Example The number of 2-subsets of a 4-element set is $\binom{4}{2}$

$$\binom{4}{2} = \frac{4!}{2!\,(4-2)!} = \frac{\cancel{4}^{2}\cdot 3\cdot \cancel{2}\cdot\cancel{1}}{\cancel{2}\cdot\cancel{1}\ \cancel{2}\cdot\cancel{1}} = \underline{\underline{6}}$$

(which agrees with our example above).

We have met the notion of an (ordered) k-tuple $(x_1, \dots, x_k)$. In what follows, $x_1, \dots x_k \in X$ some fixed set with n elements.

In general, we allow repetitions in k-tuples.

__Definition__ If $(x_1, \dots, x_k)$ is an k-tuple and $x_1, \dots x_k$ __are all different__, we say that $(x_1, \dots x_k)$ is a __k-permutation__ (over X)

__Example__ Let $X = \{a, b, c, d\}$

We list all the 2-perms over X

(a,b)	(b,a)	12
(a,c)	(b,c)	2-perms
(a,d)	(b,d)	of a
(c,a)	(d,a)	4-element
(c,b)	(d,b)	set
(c,d)	(d,c)	

An n-perm is called simply a **permutation**

Theorem 3 Let X be a set with $|X| = n$. Let $0 \le k \le n$. The number of k-perms over X is

$$ {}^nP_k = \frac{n!}{(n-k)!} $$

"n rank k"

Observe that

$$ \boxed{{}^nC_k = \frac{{}^nP_k}{k!}} $$

Example

$$ 4P_2 = \frac{4!}{2!} = \frac{4\cdot 3\cdot \cancel{2}\cdot \cancel{1}}{\cancel{2}\cdot \cancel{1}} = 12 $$

Which agrees with our example.

NB

- $^{n}C_{k}$ counts the number of k-subsets of n by choosing

- $^{n}P_{k}$ counts the number of k-perms of an n-element set by ranking

Observe that

$$^{n}P_{0} = 1$$

$$^{n}P_{n} = n!$$

all permutations

which is only

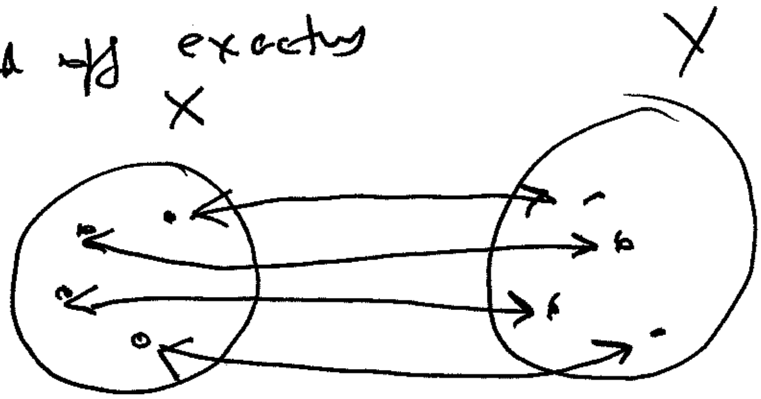
$0! = 1$

We shall prove Theorems 1, 2 and 3 but to do so we need some counting principles to help us.

Counting principles

(I) Set X and Y have the same cardinality precisely when there is a bijection between X and Y

[This means that the elements of X and Y can be paired off exactly



]

called <u>blocks of the partition</u>.

(II) Let $A = A_1 \cup \dots \cup A_n$

where (i) $A_i \neq \emptyset$ for $1 \leq i \leq n$

(ii) $A_i \cap A_j = \emptyset$ if $i \neq j$

[This is called a <u>partition of</u> A]

Then $|A| = |A_1| + \dots + |A_n|$.

(III) Let $A = A_1 \times \dots \times A_m$.

Then $|A| = |A_1| \dots |A_m|$.

(II) and (III) can be <u>proved</u> but we assume them in this course.

(I) is really a <u>definition</u> of what it means for two sets to have the same cardinality.