Lecture 6

<u>Example</u> Let $A = \{0, 1\}$

How many n-tuples of elements of A are there (called <u>binary sequences of length n</u>)

This is the set A^n so

$$|A^n| = |A|^n = \underline{\underline{2^n}}$$

<u>Example</u> The binary sequences of length 3 are

$$\left.\begin{array}{ll} (0,0,0) & (1,1,0) \\ (0,0,1) & (1,0,1) \\ (0,1,0) & (0,1,1) \\ (1,0,0) & (1,1,1) \end{array}\right\} \quad 8 = 2^3$$

<u>Proof of Theorem 1</u> We prove that $|P(A)| = 2^{|A|}$. Let $|A| = n$.

We prove that there is a bijection between

$$\{0,1\}^n \quad \text{and} \quad P(A)$$

which by (I) and our Example above proves the result.

List the elements of A as: $a_1, \dots, a_n$

A subset $B \subseteq A$ consists of choosing some elements of A (and rejecting others).

Define the binary n-tuple paired off with B to be $(e_1, \dots, e_n)$ where

End of proof

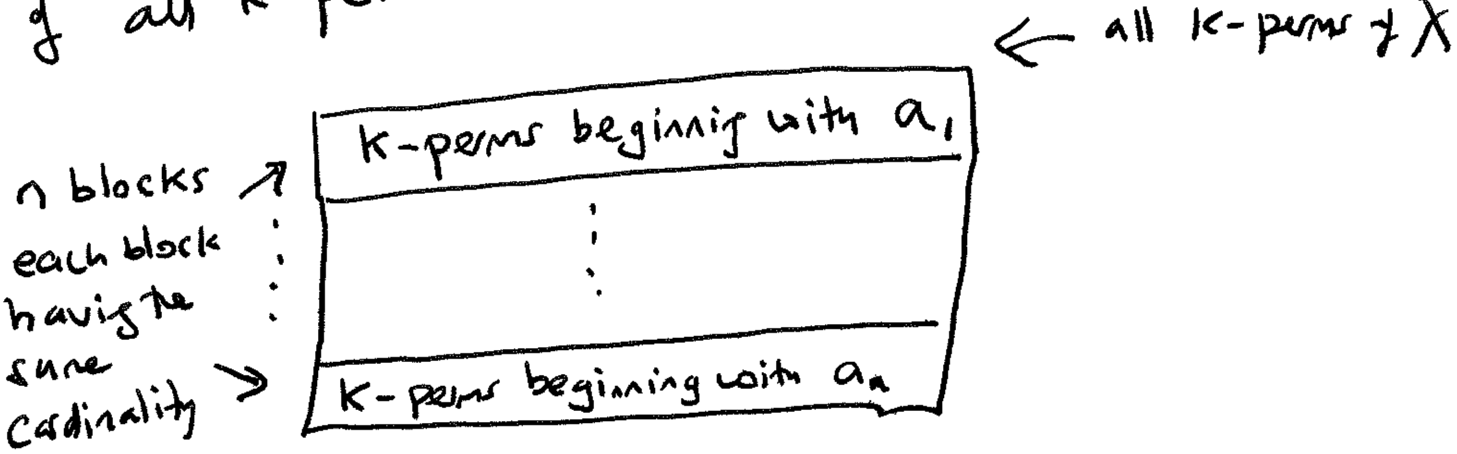
$$e_i = \begin{cases} 0 & \text{if } a_i \notin B \\ 1 & \text{if } a_i \in B \end{cases} \qquad \blacksquare$$

We prove Theorem 2 after we have proved Theorem 3.

<u>Proof of Theorem 3</u> We prove that the number of k-perms of an n-element set is

$$^{n}P_k = \frac{n!}{(n-k)!}$$
$$= n(n-)\ldots(n-(k-1))$$

Let the set $X = \{a_1, \ldots, a_n\}$. The set of all k-perms can be partitioned



By (II) $\boxed{^{n}P_k = n \, {}^{n-1}P_{k-1}}$

Now repeat...

$$^{n}P_k = n(n-1)\;{}^{n-2}P_{k-2}$$

$$= n(n-1)(n-2)\;{}^{n-3}P_{k-3}$$

$$= n(n-1)\dots(n-(k-1))\;{}^{n-k}P_0$$

But $^{n-k}P_0 = 1$ — think of this as a definition. [redacted]

Essentially, there is such a thing as ~~an~~ 0-tuple and it looks like this ( ) it is a bit like the empty set

Proof of Theorem 2

$$^{n}C_{k} = \frac{n!}{k!\,(n-k)!}$$

# of k-subsets of an n-element set.

Idea $\exists$ $k!$ k-perms
all determine the same k-subset

[$(a_1, a_2), (a_2, a_1)$ both determine $\{a_1, a_2\}$]

$$\therefore \quad ^{n}C_{k} = \frac{1}{k!}\ ^{n}P_{k} \qquad \blacksquare$$

Example

Let $X = \{a, b, c, d\}$

2-perms over X	2-subsets over X
$(a,b), (b,a)$	$\{a,b\}$
$(a,c), (c,a)$	$\{a,c\}$
$(a,d), (d,a)$	$\{a,d\}$
$(b,d), (d,b)$	$\{b,d\}$
$(b,c), (c,b)$	$\{b,c\}$
$(c,d), (d,c)$	$\{c,d\}$

This illustrates why

$$k!\,{}^nC_k = {}^nP_k$$