Lecture 7 $\boxed{\text{Section 2}}$

Section 4.3: Solving quadratic equations

This will mainly be revision but I will introduce some important terminology.

Let a, b, c be real numbers where $a \neq 0$

Then $ax^2 + bx + c$ is called a <u>quadratic polynomial</u>.

The equation

leading coefficient

$$ax^2 + bx + c = 0 \qquad (*)$$

is called a <u>quadratic equation</u>

If r is a real number (for the time being) such that

$$ar^2 + br + c = 0$$

then it is called a root of the quadratic equation.

There is a well-known formula for finding the roots of a quadratic equation.

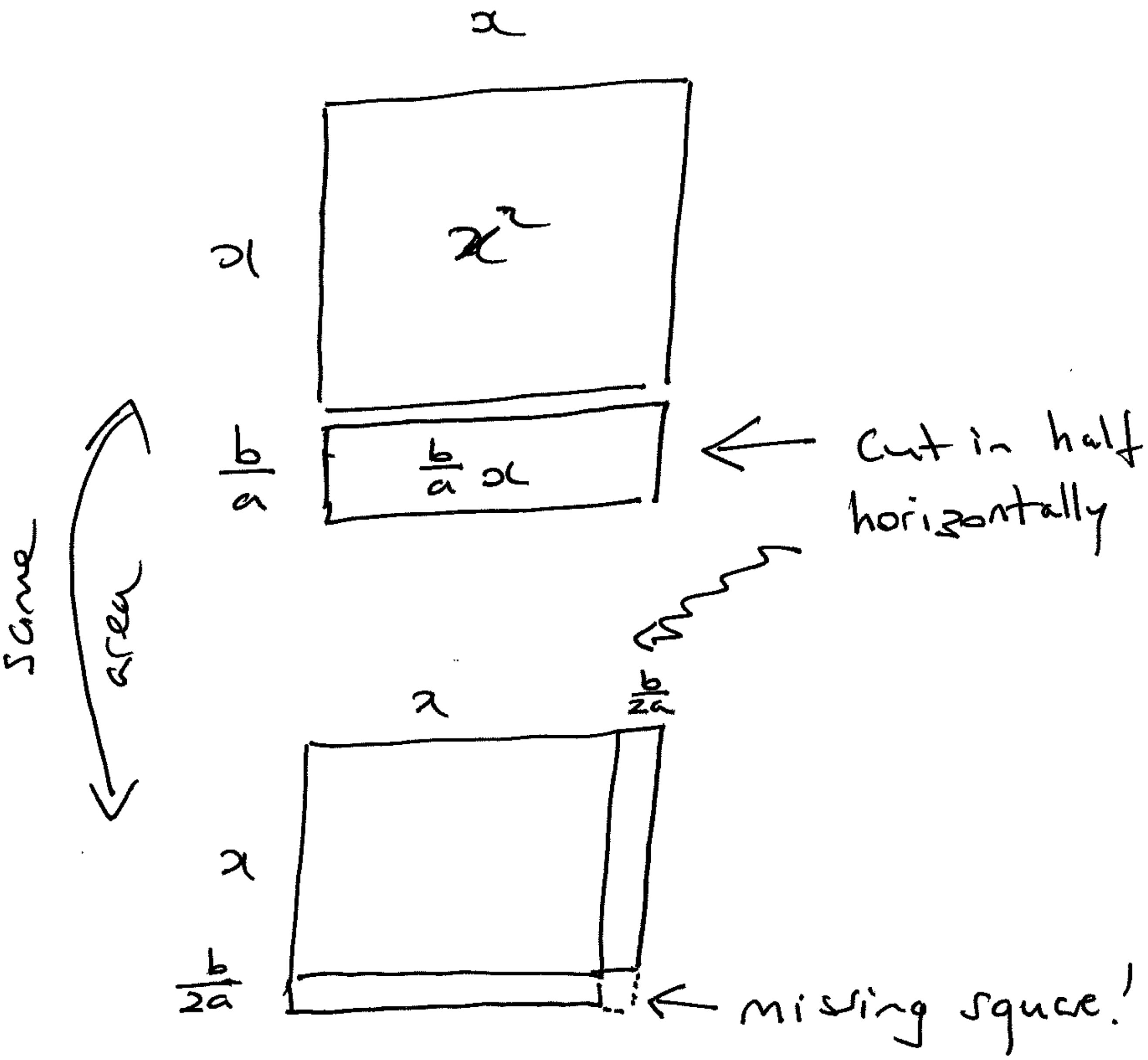
We now ~~the~~ derive that equation

Divide $(*)$ through by a (possible since $a \neq 0$!)

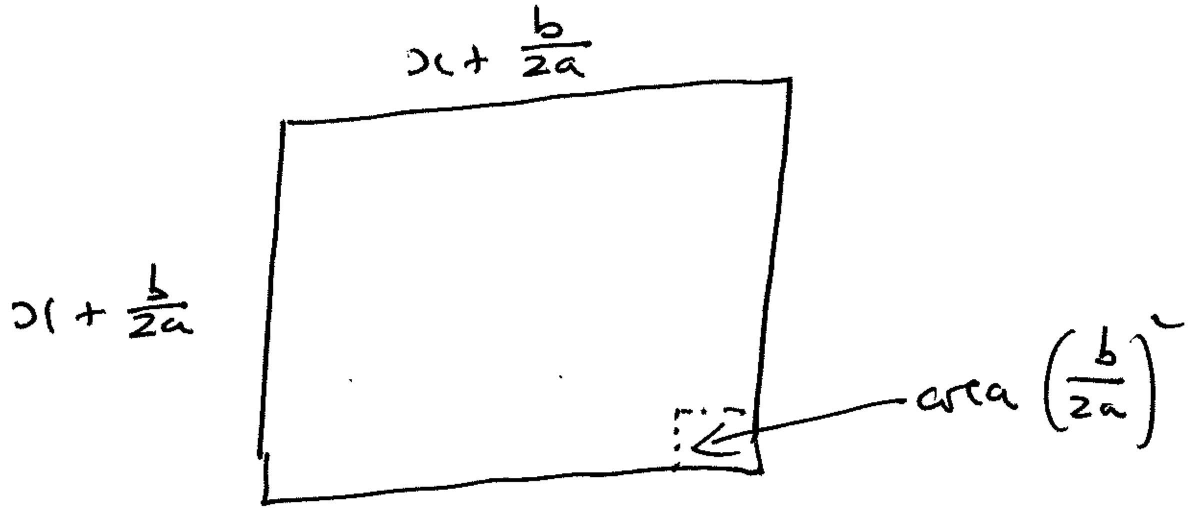
$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

monic because leading coefficient is equal to 1

We draw a picture of $x^2 + \frac{b}{a}x$ (the non-constant part of our monic quadratic)



Complete the Square



area of this square is $\left(x + \frac{b}{2a}\right)^2$

But this is bigger in area by $\left(\frac{b}{2a}\right)^2$

than the previous two pictures.

Completing the square

$$\therefore \quad \boxed{x^2 + \frac{b}{a}x = \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2}$$

Of course, this is immediate by simple algebra but the pictures explain the terminology.

The quadratic equation

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

is therefore equal to

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

This can be written as

$$\left(x + \frac{b}{2a}\right)^2 = \underbrace{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}_{\text{a number!}}$$

$$= \frac{b^2}{4a^2} - \frac{c}{a}$$

$$= \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

$$= \frac{b^2 - 4ac}{4a^2}$$

Thus $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$.

Take the square roots of both sides

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2-4ac}}{2a}$$

NB!

Therefore

$$\boxed{x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}$$

This is the formula you know and love from high-school.

Define $D = b^2 - 4ac$

called the <u>discriminant</u> of our quadratic polynomial.

There are now three cases

(1) If $\boxed{D > 0}$ then our quadratic equation has two, distinct real roots.

(2) If $\boxed{D = 0}$ then $\frac{-b}{2a}$ is one repeated root (so, still two roots but the same root counted twice)

This means that $x^2 + \frac{b}{a}x + \frac{c}{a}$ is a perfect square because

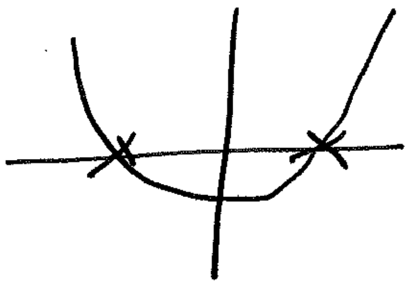
$$\left(x + \frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \frac{c}{a}$$

Check this!
Rem: $D = 0$

(3) If $\boxed{D < 0}$ then there are no real roots. We say that our quadratic polynomial is irreducible.

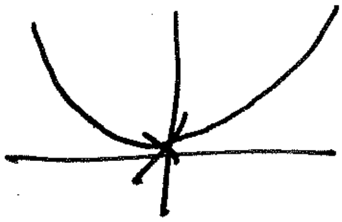
Pictures

$D > 0$



← graph of our quadratic poly (other graphs are possible — this is just illustrative)

$D = 0$



$D < 0$