Lecture 9

Example

$$(x+y)^3 = (x+y)\left[(x+y)(x+y)\right]$$

$$= (x+y)\left[(x+y)x + (x+y)y\right]$$

$$= (x+y)\left[xx + yx + xy + yy\right]$$

$$= (x+y)xx + (x+y)yx + (x+y)xy + (x+y)yy$$

$$= \left.\begin{array}{l} xxx + yxx + xyx + yyx + \\ xxy + yxy + xyy + yyy \end{array}\right\} 8 = 2^3 \text{ summands}$$

We only use distributivity & associativity

But by commutativity, some of these summands have the same value. The number of such values is counted by binomial coefficients.

$$(x+y)^3 = xxx + \boxed{yxx} + \boxed{xyx} + yyx + \boxed{xxy} + yxy + xyy + yyy$$

$\boxed{} = 3yx^2$ $\quad \bigcirc = 3y^2x$

<u>Example</u> Expand $(2a-3b)^6$ using the binomial theorem.

needs to be in the form $(x+y)^6$

$$(2a-3b)^6 = (2a+(-3b))^6$$

$$= \sum_{i=0}^{6} \binom{6}{i} \underline{(2a)^i} \, \underline{(-3b)^{6-i}}$$

$$= \sum_{i=0}^{6} \binom{6}{i} 2^i a^i \, b^{6-i} (-1)^{6-i} 3^{6-i}$$

$$= \sum_{i=0}^{6} \boxed{\binom{6}{i} (-1)^{6-i} 2^i 3^{6-i}} \, a^i b^{6-i}$$

This number is called the <u>coefficient</u> of $a^i b^{6-i}$

Calculate the coefficient of a^2b^4 in this expression.

<u>Put $i=2$</u>

Coefficient of a^2b^4 = $\binom{6}{2}(-1)^4\,2^2\,3^4$

$$= \underline{\underline{4,860}}$$

$$\boxed{\text{Careful! } (st)^n = s^n t^n}$$

Aside on exponents

Laws of exponents $a^{m+n} = a^m a^n$, $(a^m)^n = a^{mn}$. (when $m,n \geq 1$). We want these laws to persist for more general exponents.

$$a^m \cdot a^{-m} = a^{m-m} = a^0$$

But $a^m \cdot a^{-m} = \frac{a^m}{a^m} = 1$

$\therefore$ $\boxed{a^0 = 1}$

What is a^b when $a, b \in \mathbb{R}$?

~~Eg~~ Heuristic argument

$\ln(a^b) = b \ln a.$

But now we are using special functions

$\therefore$ $\boxed{a^b = \exp(b \ln a)}$

If $b = 0$ then $a^0 = \exp(0) = 1$

(What happens if $a = 0$? (or $a \to 0$?))

Complex numbers

We return to the formula for the roots of a quadratic.

$$ax^2 + bx + c = 0 \qquad a, b, c \in \mathbb{R}, \ a \neq 0.$$

The roots are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Put $D = b^2 - 4ac$, the discriminant.

Suppose $D < 0$ the real irreducible case.

$$\left.\text{Then} \quad x = \frac{-b \pm \sqrt{-|D|}}{2a}.\right\} \text{ currently meaningless}$$

Define $i = \sqrt{-1}$. So, $i^2 = -1$

Then $\boxed{x = \dfrac{-b}{2a} \pm i\sqrt{|D|}}$