1. Combinatorics

This subject is about counting. It is therefore particularly important in probability theory.

Example If we throw a dice there are 6 possible outcomes. Suppose I need to throw a 6. Exactly one of the possible outcomes is a 6. Thus the probability of throwing a 6 is $1/6$.

More generally, the probability of an event is

$$= \frac{\text{number of elements in event}}{\text{total number of possible elements}}$$

<u>Example</u> I throw two dice. What is the probability of throwing a double 6? The total number of possible elements is $6 \times 6 = 36$. The number of elements in the event 'a double 6' is 1. Thus the probability is $1/36$.

$$\boxed{\text{Elementary probability theory is mainly about counting the number of elements in a set.}}$$

Probability theory is based on sets and basic operations on sets. Thus we need to develop basic set theory first. Then we shall describe some techniques for counting the number of elements in a set (called its cardinality).

Basic set theory

See Section 3.1 of my book.

- A *set* is a collection of things, called its *elements*, which we wish to regard as a whole.
- Two sets are *equal* when they contain the same elements.

<u>Example</u> Here are some random things:

$\bigcirc$ $\uparrow$

$\square$ $\ast$

For the purposes of this lecture, I want to be able to talk about these things (i.e. regard them as a single "thing").
I therefore gather them together into a set

commas separate the elements of this set

$$\{\square, \bigcirc, \ast, \uparrow\} = A$$

I can give this set a <u>name</u>

braces/ curly brackets delimit the set

I can now talk about the set A and you know what I mean because I have told you what A is.

The <u>order</u> in which you list the elements of the set does not matter. So,

$$A = \{\uparrow, *, \circ, \square\}$$

even though the elements are listed in a different order.

A set is a bag not a display case.

The things that make up the set are called its <u>elements</u>.

We write

$\uparrow \in A$ 'is an element of'

Shorthand

We write

$\Delta \notin A$ 'is not an element'

An important feature of set notation that seems odd when first encountered is that <u>repetitions of elements are ignored</u>.

<u>Example</u> $\{1, 1\} = \{1\}$

Its elements are

$$\emptyset, \quad \{a\}, \quad \{\{a\}, \{a,b\}\}$$

The element $\{\{a\}, \{a,b\}\}$ is a set with elements $\{a\}$ and $\{a,b\}$

$$\boxed{\text{All of mathematics can be developed using sets}}$$

<u>Remark</u> See Russell's Paradox.

The set $\{\}$ has nothing in it.
It is called the <u>empty set</u>. To avoid any misunderstanding, we give this set a name: $\emptyset$.

<u>Example</u> $\emptyset \neq \{\emptyset\}$.

LHS is a set with no elements.
RHS is a set with one element.

Set notation seems simple, even simple-minded. But we can form sets whose elements are themselves sets.

<u>Example</u> $X = \{\emptyset, \{a\}, \{\{a\}, \{a, b\}\}\}$ is a set.

Let A be a set. The <u>cardinality</u> of A, written $|A|$, is simply the number of elements in A.

<u>Examples</u>

(i) $|\emptyset| = 0$.

(ii) $|\{1,2,3\}| = 3$.

(iii) $|\{a,b\}| = 2$.

Probability theory often involves calculating $|A|$ for special sets A.

I'll give you some examples later.

Some special sets with special names

$\mathbb{N} = \{0, 1, 2, \underbrace{\ldots}_{\text{read 'and so on'}}\}$ the **natural numbers.**

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

the **integers.**

$\mathbb{Q}$ all positive and negative fractions or quotients. The **rational numbers**.

$\mathbb{R}$ all real numbers

(can be written $\underset{\substack{\uparrow \\ \text{integer}}}{a} \cdot a_1 a_2 a_3 \underbrace{\ldots}_{\substack{\text{potentially} \\ \text{infinite.}}}$

Examples

(i) $-1 \notin \mathbb{N}$.

(ii) $\sqrt{2} \in \mathbb{R}$ but $\underbrace{\sqrt{2} \notin \mathbb{Q}}_{\text{This needs to be proved}}$.

(iii) $3 \in \mathbb{N}$.

(iv) $\frac{1}{2} \in \mathbb{Q}$, $2 \left(= \frac{2}{1}\right) \in \mathbb{Q}$,

$-\frac{15}{16} \in \mathbb{Q}$.

A very common way to define a set is to use a property that defines the elements of the set.

Example

$$\mathbb{P} = \{2, 3, 5, 7, 11, 13, \ldots\}$$
$$= \{a : \underbrace{a \text{ is a prime}}_{\text{property that } a \text{ has}}\}$$

$:$ such that

Examples

(i) $A = \{a : a \in \mathbb{R} \text{ and } a^2 = 1\} = \{-1, 1\}$.

(ii) $B = \{a : a \in \mathbb{N} \text{ and } a^2 = 1\} = \{1\}$.

(iii) $C = \{a : a \in \emptyset \text{ and } a^2 = 2\} = \emptyset$.

(iv) $D = \{a : a \in \mathbb{R} \text{ and } a^2 = -1\} = \emptyset$.

Let A be a set.

We can construct a set B whose elements are only chosen from A.

We then say that B <u>is a subset</u> of A, written $A \subseteq B$.

<u>Examples</u>

(i) $\emptyset \subseteq A$. Choose no elements from A!

(ii) $A \subseteq A$. Choose all elements from A.

[Maths uses the word 'Choose' in an eccentric way].