Lecture 10

Last time, I stated the binomial theorem:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Let's see if this formula makes sense by computing some special cases.

<u>$n=1$</u>

$$(x+y)^1 = \sum_{i=0}^{1} \binom{1}{i} x^i y^{1-i}$$

$$= \binom{1}{0} x^0 y^1 + \binom{1}{1} x^1 y^0$$

$$= \binom{1}{0} y + \binom{1}{1} x$$

$$= \underline{x+y}$$

$\underline{n=2}$

$$(x+y)^2 = \sum_{i=0}^{2} \binom{2}{i} x^i y^{2-i}$$

$$= \binom{2}{0} x^0 y^2 + \binom{2}{1} x^1 y^1 + \binom{2}{2} x^2$$

$$= y^2 + 2xy + x^2$$

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<u>Example</u> Expand $(2a-3b)^6$ using the binomial theorem:

$$(2a-3b)^6 = \Big(\overset{\overset{x}{\shortparallel}}{(2a)} + \overset{\overset{y}{\shortparallel}}{(-3b)} \Big)^6$$

$$= \sum_{i=0}^{6} \binom{6}{i} (2a)^i (-3b)^{6-i}$$

$$= \sum_{i=0}^{6} \binom{6}{i} 2^i a^i (-3)^{6-i} b^{6-i}$$

This is a number called the <u>coefficient</u> of $a^i b^{6-i}$

$$= \sum_{i=0}^{6} \boxed{\binom{6}{i} 2^i (-3)^{6-i}} \, a^i b^{6-i}$$

Let's now calculate the coefficient of a^2b^4 in this expansion.

We need $i = 2$.

The coefficient of a^2b^4 is therefore

$$\binom{6}{2} 2^2 (-3)^4$$

$$= \binom{6}{2} 2^2 3^4 = \underline{\underline{4,860}}$$

Careful!

$(\neq)$ $(st)^n = s^n t^n$.

Aside on exponents

- $a^m a^n = a^{m+n}$

- $(a^m)^n = a^{mn}$

- $a^{-m} = \dfrac{1}{a^m}$ (since $a^m a^{-m} = a^{m-m} = a^0 = 1$)

- $a^0 = \underline{1}$

What is a^b if $a, b \in \mathbb{R}$

(For example, what does $\sqrt{2}^{\pi}$ mean).

Let $r = a^b$.

then $\ln(r) = b \ln(a)$

So $\exp(\ln(r)) = r$

$\therefore \boxed{a^b = e^{b \ln(a)}}$

Complex numbers

Definition The symbol i is defined to have the property that $i^2 = -1$.

Definition A **complex number** is an entity of the form $a + ib$ (or $a + bi$) where $a, b \in \mathbb{R}$. $a + ib = c + id$ $\iff$ $a = c$, $b = d$.

The set of complex numbers is denoted by $\mathbb{C}$.

Observe that $\mathbb{R} \subseteq \mathbb{C}$ since

$r = r + 0i$. Thus every real number is a (special kind of) complex number.

We assume that $\mathbb{C}$ is also a field.

[All of my claims can be justified).

Addition of complex numbers

$$(a+ib) + (c+id) = a+ib + c+id$$
$$= (a+c) + i(b+d).$$

Multiplication of complex numbers

$$(a+ib)(c+id) = ac + iad + ibc + i^2 bd$$

$$= (ac - bd) + i(ad+bc).$$

Division (except by zero)

This requires a trick.

$$\frac{1}{a+ib} = \frac{a-ib}{(a+ib)(a-ib)}.$$

We call $a-ib$ the <u>complex conjugate</u> of $a+ib$

Observe that

$$(a+ib)(a-ib) = a^2 - iab + iba - i^2b^2$$
$$= \underline{\underline{a^2+b^2}}$$

$$\therefore \frac{1}{a+ib} = \frac{1}{a^2+b^2}\Big(a-ib\Big)$$

If $z \in \mathbb{C}$ then $\bar{z}$ is its complex conjugate.

Example Every quadratic equation which is irreducible has two complex roots.

Given $ax^2 + bx + c = 0$ (where $a, b, c \in \mathbb{R}$, $a \neq 0$)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Suppose that $b^2 - 4ac = D < 0$.

Then the roots of the quadratic equation are

$$x = \frac{-b \pm i\sqrt{|D|}}{2a}$$ ← absolute value of D