Lecture 11

Example Calculate the roots of

$$x^2 - 2x + 2 = 0.$$

By the quadratic formula

$$x = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm \sqrt{-4}}{2}$$

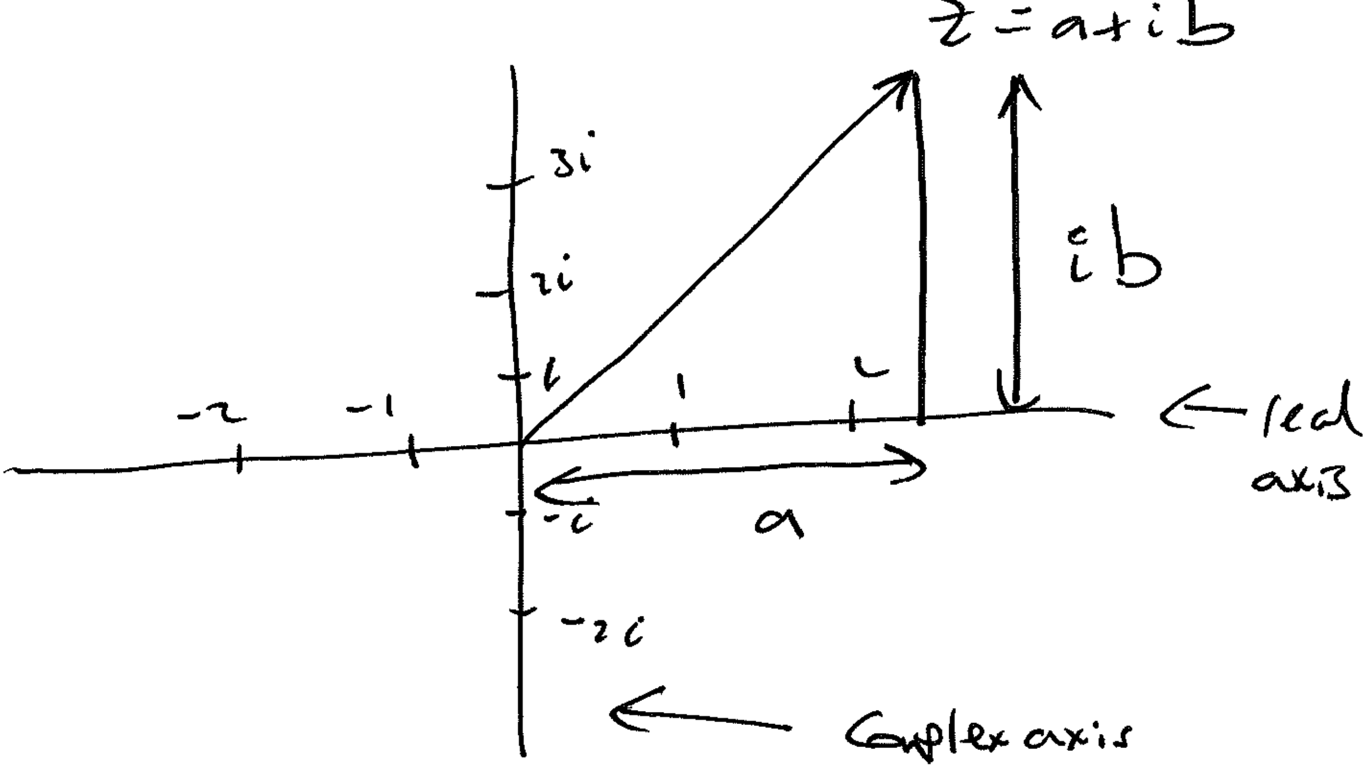
$$= \frac{2 \pm 2i}{2}$$

$$= \underline{\underline{1 \pm i}}$$

Check $(1+i)^2 - 2(1+i) + 2 = 0$ ✓

$(1-i)^2 - 2(1-i) + 2 = 0$ ✓

A complex number is $a+bi$ so it is irresistable to plot complex numbers in the <u>complex plane</u>



The <u>length</u> of the <u>hypotenuse</u> is $\sqrt{a^2+b^2} = \sqrt{z\bar{z}}$

This is a real number.

We write $|z| = \sqrt{a^2+b^2}$ called the <u>modulus</u> of z.

Lemma $|uv| = |u|\,|v|$

Proof Let $u = a+bi$ and $v = c+di$.

Then $|u| = \sqrt{a^2+b^2}$

$|v| = \sqrt{c^2+d^2}$

$\therefore |u|\,|v| = \sqrt{(a^2+b^2)(c^2+d^2)}$

$$uv = (a+bi)(c+di)$$
$$= ac + adi + bci - bd$$
$$= (ac-bd) + (ad+bc)i$$

So,

$$|uv| = \sqrt{(ac-bd)^2 + (ad+bc)^2}$$

$$= \sqrt{(ac)^2 - \cancel{2(ac)(bd)} + (bd)^2 + (ad)^2 + \cancel{2(adbc)} + (bc)^2}$$

$$= \sqrt{(ac)^2 + (bd)^2 + (ad)^2 + (bc)^2}$$

$$= \sqrt{(a^2+b^2)(c^2+d^2)}$$

$\therefore |uv| = |u||v|$ as claimed $\blacksquare$

<u>Theorem</u> Every non-zero complex number has ~~exactly~~ two square roots.

<u>Proof</u> <u>Input</u>: $a + bi$

<u>Output</u>: $x + yi, \ -(x+yi)$

such that $(x+yi)^2 = a+bi$

Let $(x + iy)^2 = a + bi \quad (*)$

$(x+iy)^2 = (x^2 - y^2) + 2xyi$

Compare real and imaginary parts:

(1) $x^2 - y^2 = a.$

(2) $2xy = b$

We shall now find a third equation that will make solving this problem almost trivial.

Since $(x+iy)^2 = a+bi$

We must have that

$$|(x+iy)^2| = |a+bi| = \sqrt{a^2+b^2}$$

But $|(x+iy)^2| = |x+iy|\,|x+iy|$ by Lemma

$$= x^2+y^2$$

$\therefore$ (3) $x^2+y^2 = \sqrt{a^2+b^2}$

We therefore have the following 3 equations:

(1) $x^2 - y^2 = a.$

(2) $2xy = b.$

(3) $x^2 + y^2 = \sqrt{a^2 + b^2}$

(1) + (3) gives

$$2x^2 = a + \sqrt{a^2 + b^2}$$

$$\therefore \quad x^2 = \frac{1}{2}\left[a + \sqrt{a^2 + b^2}\right]$$

$$\therefore \quad x = \pm\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}$$

For each of these two values of x, (2) delivers two values of y. $\blacksquare$

Example Find the square roots of $3+4i$.

Let

$$(x+iy)^2 = 3+4i$$

& $(x^2-y^2) + 2xyi = 3+4i$

$\therefore$ (1) $x^2-y^2 = 3$.

(2) $2xy = 4$.

(3) $x^2+y^2 = \sqrt{3^2+4^2}$

$= \sqrt{9+16}$

$= \sqrt{25} = 5$.

(1) + (3) gives $2x^2 = 8 \therefore x^2 = 4$

& $\underline{\underline{x = \pm 2}}$

Now use equation (2) to get corresponding values of y

$$x = 2 \,, \quad y = \frac{4}{2x} = \frac{4}{4} = \underline{1}$$

$$x = -2 \,, \quad y = \frac{4}{-4} = \underline{-1}$$

$\therefore$ Square roots of $3+4i$ are

$\underline{2+i}$ and $\underline{-(2+i)}$

Check $\quad (2+i)^2 = 4 + 4i - 1$

$$= \underline{\underline{3+4i}} \checkmark$$