Lecture 12

Example Solve the following quadratic equation.

$$x^2 + 4ix + (4i-1) = 0.$$

By the usual formula (which works because $\mathbb{C}$ is a field in which all square roots exist):

$$x = \frac{-4i \pm \sqrt{(4i)^2 - 4(4i-1)}}{2}$$

$$= \frac{-4i \pm 2i\sqrt{3+4i}}{2}$$

$$= -2i \pm i(2+i) \leftarrow \text{by a previous calculation}$$

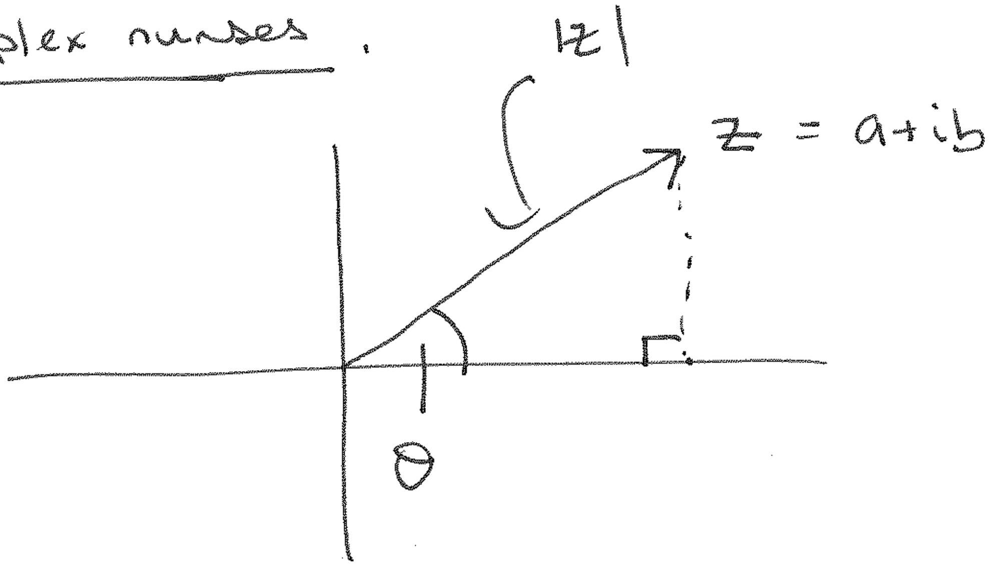
$$= \underline{\underline{-1 \text{ or } -4i+1}}$$

Check

$$(-1)^2 + 4i(-1) + (4i-1) = 1 - 4i + 4i - 1 = 0 \checkmark$$

$$(-4i+1)^2 + 4i(-4i+1) + (4i-1) = 0 \checkmark$$

What we have seen so far can be regarded as the arithmetic of complex numbers. To make progress, however, we shall also need to develop the geometry of complex numbers.



Let $z \neq 0$.

Then $z = |z|(\cos\theta + i\sin\theta)$

by basic trig. This is called the polar form of the complex number z. The angle θ or more accurately the angles $\theta + 2\pi n$ (where $n \in \mathbb{Z}$) is called / are called the argument / arguments of z.

The polar form is therefore not unique.

Multiplication of complex numbers has a natural interpretation when viewed <u>geometrically</u>.

<u>Proposition</u> Let

$$u = |u| \ (\cos\theta + i\sin\theta)$$

and

$$v = |v| \ (\cos\phi + i\sin\phi)$$

Then $uv = |u||v| \left(\cos(\theta+\varphi) + i\sin(\theta+\varphi) \right)$

i.e. multiply the moduli and add the arguments.

<u>Proof</u> By basic trig:

$$uv = |u| \left(\cos\theta + i\sin\theta\right) |v| \left(\cos\phi + i\sin\phi\right)$$

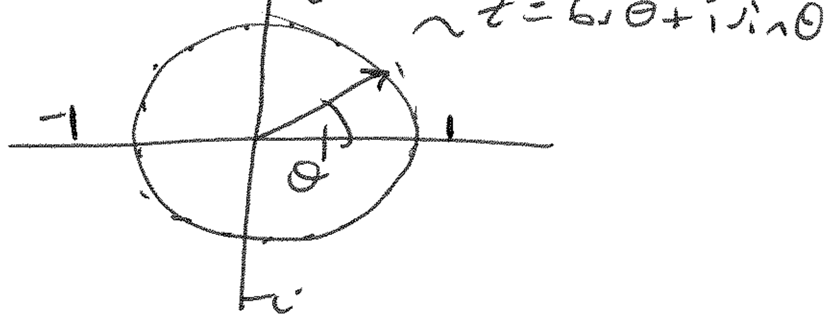
$$= |u||v| \left[(\cos\theta\cos\phi - \sin\theta\sin\phi) + i\,(\cos\theta\sin\phi + \sin\theta\cos\phi) \right]$$

$$= |u||v| \left(\cos(\theta+\varphi) + i\sin(\theta+\varphi) \right) \blacksquare$$

<u>Example</u> $i \times (\ )$ multiplying by i rotates a complex number by 90° in an anticlockwise direction. Thus $i^2 \times (\ )$ rotates a complex number by 180° in an anticlockwise direction. But this is equal to $-1 \times (\ )$. This gives us a way to picture the sense in which $i^2 = -1$.

We can now make the following deductions from the proposition above.

- Complex numbers of modulus 1 lie on the circumference of the unit circle in the complex plane.



Let $u = \cos\theta + i\sin\theta.$

Then $u^2 = \cos 2\theta + i\sin 2\theta.$

& $u^3 = \cos 3\theta + i\sin 3\theta.$

More generally, we have proved what is often called

<u>De Moivre's theorem</u>

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

where $n \in \mathbb{N}$.

Complex numbers "know" about trig identities

<u>Example</u> By De Moivre

(1) $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta.$

But (by the binomial theorem in general)

(2) $(\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta$

$= (\cos^2\theta - \sin^2\theta) + 2\cos\theta\sin\theta\, i$

Now compare (1) and (2) real and imaginary parts:

- $\cos 2\theta = \cos^2\theta - \sin^2\theta$
- $\sin 2\theta = 2\cos\theta\sin\theta.$

This example can be easily generalized.