Lecture 13

Euler's formula

Assume that, we can write

$$(*) \quad e^x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

Need to find $a_0, a_1, a_2, a_3, \dots$

Put $x = 0$ $\quad \underline{1} = a_0$.

Now differentiate both sides of $(*)$

$$(**) \quad e^x = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

Put $x = 0$ $\quad \underline{1} = a_1$

Now differentiate both sides of $(**)$ to get

$$e^x = 2a_2 + 3 \cdot 2 \cdot a_3 x + \dots$$

Put $x = 0$ get $\quad a_2 = 1/2$

Continuing in this way, we find that

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots.$$

We now do exactly the same things for $\sin x$ and $\cos x$ (Rem x is in radians)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

We calculate $e^{i\theta}$

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots$$

i	i^2	i^3	i^4	i^5
i	-1	$-i$	1	i

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{\theta^3}{3!}i + \frac{\theta^4}{4!} -$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \right) +$$

$$i\left(\theta - \frac{\theta^3}{3!} + \dots\right)$$

$$\therefore \boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

This formula links, sin, cos, exp via i

Put $\theta = \pi$

Euler's Formula

$$e^{i\pi} = -1$$

We want to calculate cube roots, fourth roots, etc of complex numbers, but before we can do that we need some material on polynomials.

Polynomials

A *polynomial of degree n* with complex coefficients looks like this

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where $a_n \neq 0$, and $a_n, \dots, a_0 \in \mathbb{C}$.

A *root* of $p(x)$ is a number $r \in \mathbb{C}$ s.t. $p(r) = 0$. We write *$\deg p(x)$* for the *degree of $p(x)$*. If $p(x)$ is identically zero we do not define its degree. If $a_n = 1$ the polynomial is called *monic*

The degree of a polynomial is a measure of how complicated it is.

Terminology

Degree	name
1	linear
2	quadratic
3	cubic
4	quartic
5	quintic

Why are we interested in finding the roots of a poly?

Let $f(x)$, $g(x)$ be polynomials

If $g(x) = f(x)\,q(x)$ for some poly $q(x)$

we say that $f(x)$ divides $g(x)$ written $f(x) \mid g(x)$.

We call $q(x)$ the quotient.

More generally, we have the following.

<u>Remainder Theorem</u> Let $f(x)$ and $g(x)$ be polynomials. <u>Either</u>

(1) $\quad g(x) \mid f(x)$

or

(2) $\quad f(x) = g(x)\,q(x) + r(x)$ ← the <u>remainder</u>

where $\deg r(x) < \deg g(x)$

<u>Remark</u> You should know from school how to divide by linear polynomials. See my book (page 181) for general polynomial long division.