Lecture 14

Notation

$\mathbb{C}[x]$ polynomials with $\mathbb{C}$ coefficients.

$\mathbb{R}[x]$ ———————— $\mathbb{R}$ ———— .

$\mathbb{Q}[x]$ ———————— $\mathbb{Q}$ ———— .

Lemma Let $f(x)$ and $g(x)$ be two non-zero polynomials. Then $f(x)g(x)$ is a non-zero polynomial and

$$\boxed{\deg f(x)g(x) = \deg f(x) + \deg g(x)}$$

Proof Let $f(x) = a_m x^m + \text{stuff}$, $g(x) = b_n x^n + \text{stuff}$. Then

lower degree terms.

$$f(x)g(x) = a_m b_n x^{m+n} + \text{stuff}$$

∎

<u>Lemma</u> (Key result) Let $f(x)$ be a polynomial.

Then $f(a) = 0 \iff (x-a) \mid f(x)$.

a is a <u>root</u> (a number)

$x - a$ is a <u>linear polynomial</u> (~~not~~ a number)

i.e. roots correspond exactly to linear factors

<u>Proof</u> <u>Easy direction first</u>

s'pose $(x-a) \mid f(x)$.

Then $f(x) = (x-a)\, q(x)$ for some $q(x)$.

Put $x = a$.

Then RHS $= 0$ so LHS $= 0$.

It follows that $f(a) = 0$ and so a is a root of $f(x)$. $\square$

<u>Hard direction</u> Suppose that $f(a) = 0$.

There are two possibilities:

(1) $(x-a) \mid f(x)$.

(2) $(x-a) \nmid f(x)$.

We show that (2) cannot happen.

$$f(x) = (x-a)\, q(x) + r(x)$$

where $\deg r(x) < \deg (x-a)$.

It follows that $r(x) = r$, an <u>non-zero</u> constant. Now

$$0 = f(a) = 0 + r$$

So $0 = r$ # Contradiction. ∎

This correspondence between roots and linear factors enables us to precisely define the number of roots a polynomial has.

Counting roots

Let $p(x)$ be a poly. Suppose that $(x-a)^m \mid p(x)$ but $(x-a)^{m+1} \nmid p(x)$. Then we say that a is a root of $p(x)$ with <u>multiplicity m</u>.

We always count roots according to their multiplicity.

<u>Example</u> Consider the polynomial

$ax^2 + bx + c$ (where $a \neq 0$).

Suppose that $D = b^2 - 4ac = 0$.

Then

$$x^2 + \frac{b}{a}x + \frac{c}{a} = \left(x + \frac{b}{2a}\right)^2$$

Thus $-\frac{b}{2a}$ is a root of $ax^2 + bx + c$

with multiplicity 2.

<u>Theorem</u> A non-constant polynomial of degree n has <u>at most</u> n roots.

<u>Proof</u> Let $\deg f(x) = n$.

If $f(x)$ has a root a_1 then

$$f(x) = (x - a_1) f_1(x)$$

where $\deg f_1(x) = n-1$

Thus the maximum number of roots a poly of degree n can have is n $\blacksquare$

Example

Every real quadratic poly

$$ax^2 + bx + c$$

has exactly two roots (counting multiplicity)

- $D > 0$. Then 2 real roots (distinct)
- $D = 0$. One real root of multiplicity 2.
- $D < 0$. Two complex roots.