Lecture 15

n^{th} roots of complex numbers

Let $a \in \mathbb{C}$, $a \neq 0$.
We prove that $z^n - a = 0$ has exactly n roots.

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n^{th} roots of unity

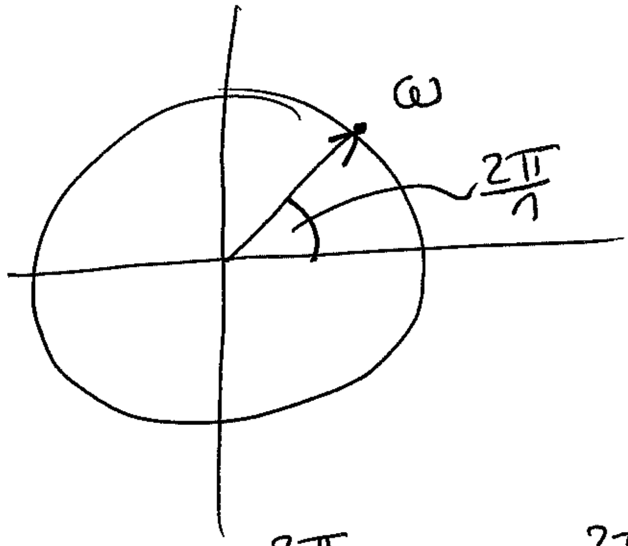
We find all the roots of the equation $x^n - 1 = 0$

Examples

(1) The roots of $x^2 - 1$ are ± 1.

(2) The roots of $x^4 - 1$ are $\pm 1, \pm i$.

To find all n^{th} roots of unity, we first locate a special n^{th} root (not 1!) called the principal n^{th} root



Define $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

By properties of complex multiplication or De Moivre's theorem, $\omega^n = \cos n\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$

$= \underline{1}$.

We call ω the principal n^{th} root of unity

Consider now the following n complex numbers:

$$\underbrace{\omega, \omega^2, \omega^3, \ldots, \omega^{n-1}, \omega^n = 1}$$

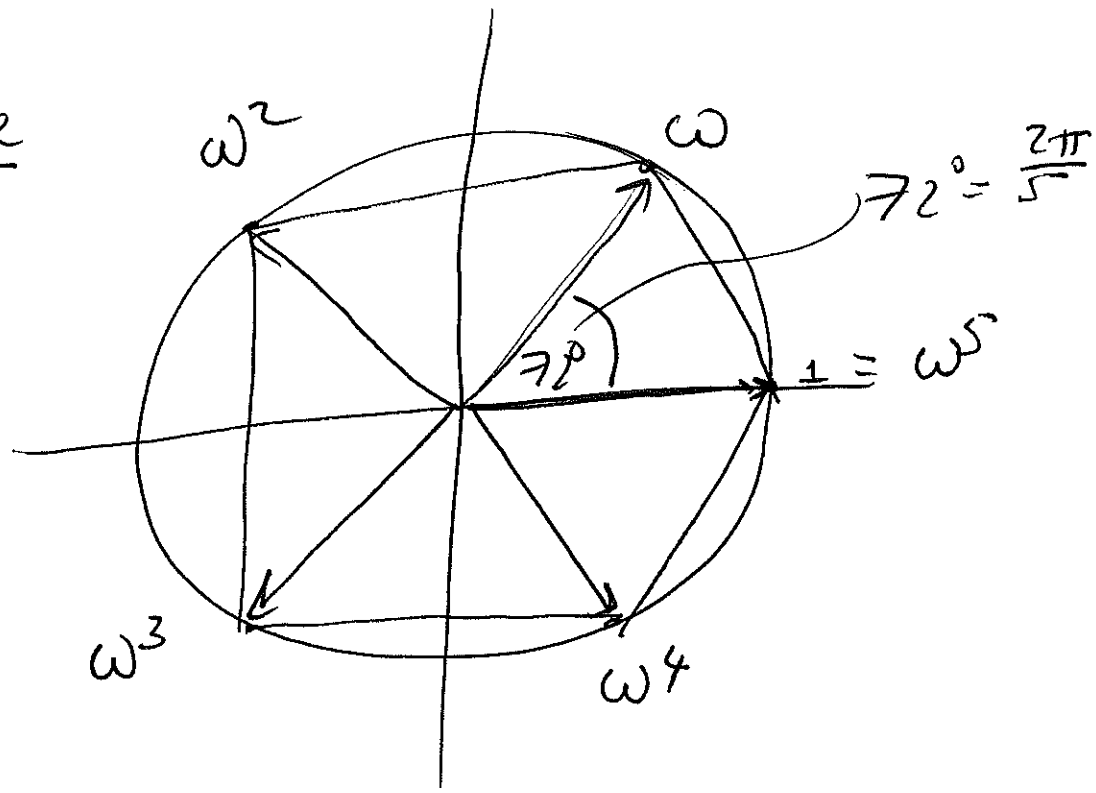
These are all the n^{th} roots of unity.

- Let $1 \le j \le n$. Then

$$(\omega^j)^n = \omega^{jn} = (\omega^n)^j = 1^j = 1$$

- The numbers $\omega, \omega^2, \dots, \omega^n$ are all different since their arguments are all different.

<u>Example</u>



$1, \omega, \omega^2, \omega^3, \omega^4$ form the vertices of a regular polygon.

We can use these polar forms to calculate n^{th} roots to any degree of accuracy

<u>Example</u> The fifth roots of unity

$$\omega \approx 0.309 + 0.9511 i$$

$$\omega^2 \approx -0.809 + 0.5878 i$$

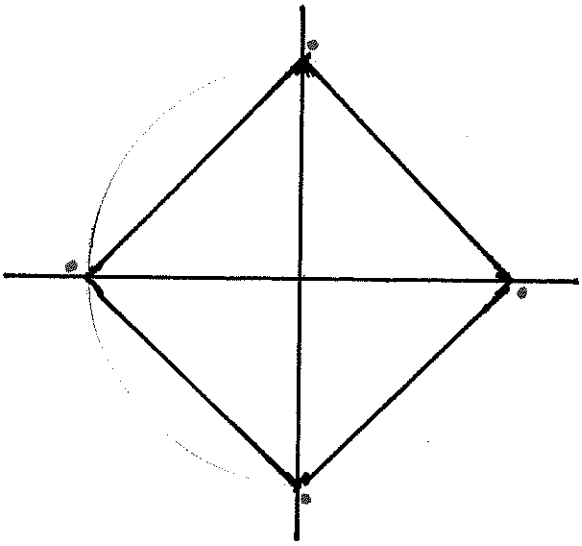
$$\omega^3 \approx -0.809 - 0.5878 i$$

$$\omega^4 \approx 0.309 - 0.9511 i$$

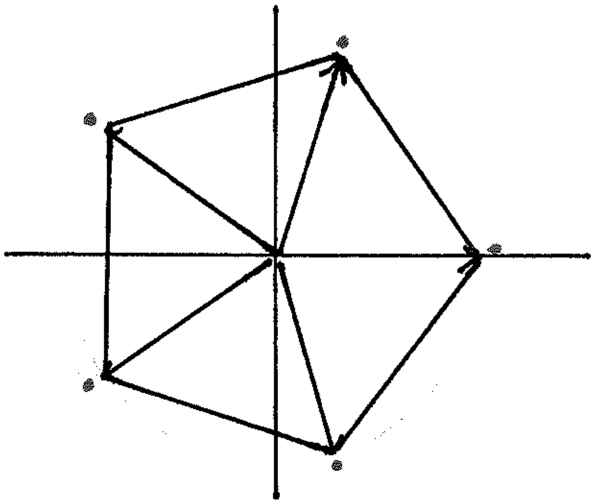
$$\omega^5 \approx 1.000 + 0.0001 i \approx \underline{1}$$

However, this might be OK in physics or engineering, but in Maths we want <u>exact</u> solutions.

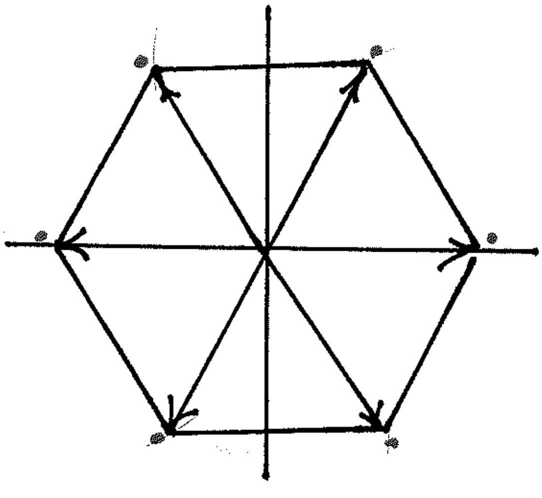
4th roots of unity



5th roots of unity



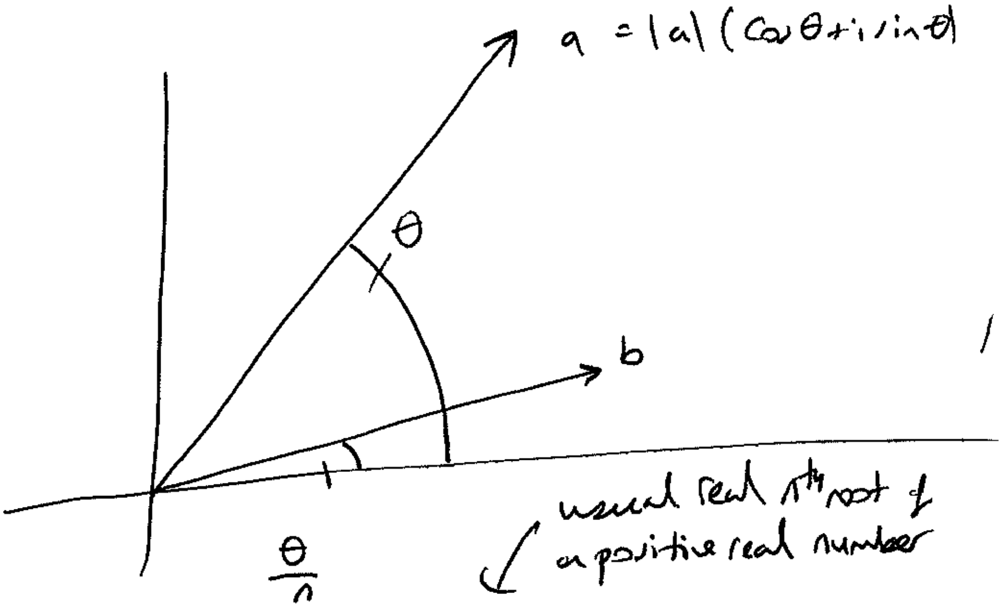
6th roots of unity



Arbitrary n^{th} roots

Let a be a complex no. $(a \neq 0)$.

Write $a = |a|(\cos\theta + i\sin\theta)$ polar form.



Define $b = \sqrt[n]{|b|}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$

Then $b^n = a$ is called the principal n^{th} root of a

Let $\omega, \omega^2, \ldots, \omega^{n-1}, \omega^n = 1$ be the n^{th} roots of unity. Then the n^{th} roots of a are:

$$b\omega,\ b\omega^2, \ldots, \omega^{n-1}b,\ b$$

<u>Example</u> What are the 4^{th} roots of 2.

The principal 4^{th} root is $\sqrt[4]{2}$.

Thus the 4^{th} roots of 2 are:

$$i\sqrt[4]{2}, \quad -i\sqrt[4]{2}, \quad -\sqrt[4]{2}, \quad \sqrt[4]{2}$$

since $i, -i, -1, 1$ are the 4^{th} roots of unity.

Radical expressions (not politics)

Roughly speaking, a radical expression is a complex number where n^{th} roots (= radicals) are allowed but no sines or cosines

Example The principal 3^{rd} root of 1 is

$$\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$ (trig form not a radical expression)

but we can also explicitly write down what $\cos\frac{2\pi}{3}$ and $\sin\frac{2\pi}{3}$ in terms of roots to get

$$\omega = \frac{1}{2}\left(-1 + i\sqrt{3}\right)$$ (radical expression).

$$\Big[\ \omega^2 = \frac{1}{4}\left(1 - 2i\sqrt{3} - 3\right)$$

$$= \frac{1}{4}\left(-2 - 2i\sqrt{3}\right)$$

$$= -\frac{1}{2}\left(1 + i\sqrt{3}\right)$$

$$\omega^3 = \frac{1}{2}\left(-1 + i\sqrt{3}\right)\cdot -\frac{1}{2}\left(1 + i\sqrt{3}\right)$$

$$= -\frac{1}{4}\left(-1 - \cancel{i\sqrt{3}} + \cancel{i\sqrt{3}} - 3\right) = 1\ \Big]$$

We have seen that $x^n - a = 0$ has exactly n roots when $a \neq 0$.
We now generalize...

The following will not be proved

Theorem (The fundamental theorem of algebra)

Every non-constant polynomial has a root.

Corollary Let $p(x)$ be a poly of degree n.

Then $p(x)$ has exactly n roots and we can write

$$p(x) = a(x - a_1) \cdots (x - a_n)$$

where $a, a_1, \ldots, a_n \in \mathbb{C}$.

We now apply the above result to real polynomials