Lecture 16

We now apply the results we have for Complex polynomials to real polynomials.

$$\boxed{\begin{aligned} &\text{If } z = a+ib \text{ then} \\ &\bar{z} = a-ib \end{aligned}}$$

Lemma

(1) $\overline{z_1 + \dots + z_n} = \bar{z}_1 + \dots + \bar{z}_n$

(2) $\overline{z_1 \dots z_n} = \bar{z}_1 \dots \bar{z}_n$.

(3) $z \in \mathbb{R} \iff z = \bar{z}$.

proofs left as exercises

Lemma Let $p(x) \in \mathbb{R}[x]$ ← set of all real polynomials.

If z is a root of $p(x)$ then $\bar{z}$ is a root of $p(x)$.

Proof Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

where $a_0, a_1, \dots a_n \in \mathbb{R}$.

Let z be a root of $p(x)$.

Then $p(z) = 0$.

It follows that

$$a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 = 0.$$

Apply complex conjugation to both sides.

$$\overline{a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0} = 0$$

Now use the results of the Lemma to get

$$a_n \bar{z}^n + a_{n-1} \bar{z}^{n-1} + \dots + a_1 \bar{z} + a_0 = 0.$$

It follows that $\bar{z}$ is a root of $p(x)$. ∎

<u>Lemma</u> Let $z \in \mathbb{C} \setminus \mathbb{R}$.

Then $(x-z)(x-\bar{z})$ is a real irreducible quadratic polynomial.

<u>Proof</u> $(x-z)(x-\bar{z}) =$

$$x^2 - x\bar{z} - zx + z\bar{z} = x^2 - x(z+\bar{z}) + z\bar{z}$$

Put $z = a+ib$. Then

- $z+\bar{z} = a+ib + a - ib = \underline{\underline{2a}}$
- $z\bar{z} = a^2+b^2$. $\therefore$ $x^2 - x(z+\bar{z}) + z\bar{z} = x^2 - 2ax + a^2+b^2$

It follows that $(x-z)(x-\bar{z})$ <u>is</u> a real quadratic. We now calculate the discriminant of $x^2 - 2ax + a^2+b^2$

$$D = (-2a)^2 - 4(a^2+b^2)$$

$$= 4a^2 - 4a^2 - 4b^2 = -4b^2 < 0$$

$\therefore$ quadratic is irreducible ∎

<u>Theorem</u> [Fundamental theorem of real polynomials]

Every non-constant real polynomial can be written as a product of <u>real linear</u> and <u>real irreducible quadratic</u> polynomials.

<u>Proof</u> Let $p(x) \in \mathbb{R}[x]$ ← degree n.

Divide the n roots of $p(x)$ into

- $r_1, \ldots, r_s \in \mathbb{R}$
- $z_1 \ldots z_t \in \mathbb{C} \setminus \mathbb{R}$

But the complex roots come in complex conjugate pairs

$u_1, \overline{u_1}, \ldots, u_m, \overline{u_m}$.

$$\therefore \; p(x) = a(x - r_1)\ldots(x - r_s)\overbrace{(x - u_1)(x - \overline{u_1})} \ldots \overbrace{(x - u_m)(x - \overline{u_m})}$$

real const.

but $(x - u_j)(x - \overline{u_j})$ is a real irreducible quadratic ∎

<u>Example</u> Let

$$p(x) = x^4 - 3x^3 + 3x^2 - 3x + 2.$$

This is a real polynomial. It can be written

$$p(x) = (x-1)(x-2)(x^2+1)$$

where x^2+1 is a real irreducible quadratic since $D < 0$.

How to find roots

(1) If $p(x) \in \mathbb{R}[x]$ and you are given a root $u \in \mathbb{C} \setminus \mathbb{R}$ then $\bar{u}$ will also be a root.

(2) Let $p(x)$ be monic with integer coefficients. Then any integer root must divide the constant term of $p(x)$.

Proof $\quad p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$

monic

$a_0, a_1, \dots a_{n-1} \in \mathbb{Z}$.

Let $s \in \mathbb{Z}$ be a root. Then

$$0 = s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0$$

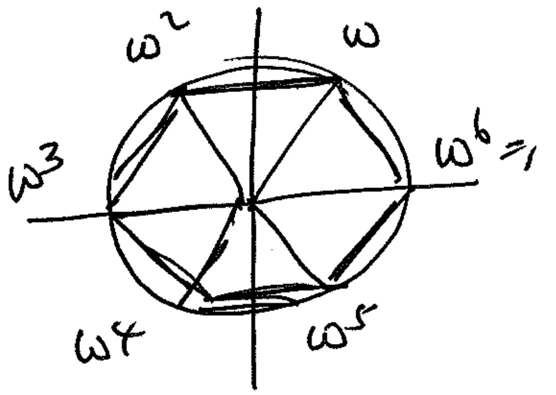
$$\therefore a_0 = -s\underbrace{(s^{n-1} + a_{n-1}s^{n-2} + \dots + a_1)}_{\text{integer}}$$

$\therefore s \mid a_0$ ∎

Example We ~~need~~ calculate the factorization of $x^6 - 1$.

Let ω be the principal 6^{th} root of unity. Then the roots of unity are $\omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6 = 1$.

$$\therefore\ x^6 - 1 = (x-\omega)(x-\omega^2)(x-\omega^3)(x-\omega^4)(x-x^5)(x-1)$$



$$x^6 - 1 = (x-1)(x-\omega^3)\overbrace{(x-\omega)(x-\omega^5)}\overbrace{(x-\omega^2)(x-\omega^4)}$$

$$= (x-1)(x+1)\overbrace{(x-\omega)(x-\omega^5)}\overbrace{(x-\omega^2)(x-\omega^4)}$$

In this case, we can express ~~roots~~ 6th roots of unity very easily in radical form.

$$\begin{aligned}
\omega &= \tfrac{1}{2}\left(1 + i\sqrt{3}\right)\\
\omega^2 &= \tfrac{1}{2}\left(-1 + i\sqrt{3}\right)\\
\omega^3 &= -1\\
\omega^4 &= -\tfrac{1}{2}\left(1 + i\sqrt{3}\right)\\
\omega^5 &= -\tfrac{1}{2}\left(-1 + i\sqrt{3}\right)\\
\omega^6 &= 1
\end{aligned}$$

$$\omega + \omega^5 = 1$$
$$\omega^2 + \omega^4 = -1$$

using these values we get that

$$x^6 - 1 = (x-1)(x+1)(x^2 - x + 1)(x^2 + x + 1)$$

both irreducible quadratics