Lecture 17

Revision of Section 2

1. Write $\dfrac{5+6i}{7+8i}$ in the form $a+bi$ where $a, b \in \mathbb{R}$.

2. Find the square roots of $-125+300i$ and show that your solutions work

3. Find all 8^{th} roots of unity in (i) Trig form (ii) Radical form.

4. Express $\cos 10\theta$ in terms of a sum of powers of $\cos\theta$ and $\sin\theta$.

5. Write the following real polynomial as a product of real linear and real irreducible quadratic polynomials (you should show explicitly that your quadratics are irreducible).

$$p(x) = x^7 - 6x^6 + 16x^5 - 36x^4 + 59x^3 - 54x^2 + 44x - 24$$

<u>Clue</u>: i is a root

Solutions

1.

$$\frac{5+6i}{7+8i} = \frac{(5+6i)(7-8i)}{(7+8i)(7-8i)}$$

Complex conjugate of $7+8i$

$$= \frac{1}{49+64}\left[(5+6i)(7-8i)\right]$$

$$= \frac{1}{113}(83+2i)$$

$$= \underline{\underline{\frac{83}{113} + \frac{2}{113}i}}$$

2. Let $(x+iy)^2 = -125+300i$

Then $(x^2-y^2)+2ixy = -125+300i$

Equate real and imaginary parts

① $x^2-y^2 = -125$

② $xy = 150$

To get equation ③ Modulus both sides

③ $x^2 + y^2 = \sqrt{125^2 + 300^2}$
$= 325.$

① + ③ yields

$2x^2 = -125 + 325$
$= 200$

$\therefore x^2 = 100$

$\therefore x = \pm 10$

Now use ②

When $x = 10$, $y = 15$

When $x = -10$, $y = -15$

$\therefore$ Square roots of $-125 + 300i$ are

$10 + 15i$ and $-(10 + 15i)$

check $(10 + 15i)^2 = -125 + 300i$

3. (i) The principal 8th root of unity is

Trig Form

$$\boxed{\omega = \cos\frac{2\pi}{8} + i\sin\frac{2\pi}{8}.}$$

$\therefore$ The 8th roots of unity are

$$\omega,\ \omega^2,\ \omega^3,\ \omega^4,\ \omega^5,\ \omega^6,\ \omega^7,\ \omega^8 = 1$$

(ii) We need to express ω in terms of radicals.
$$\frac{2\pi}{8} = \frac{360}{8} = 45^\circ$$

$$\sin 45^\circ = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}}$$

Radical Form

$$\therefore\ \boxed{\omega = \frac{1}{\sqrt{2}}(1 + i)}$$

<u>Check</u> I show now that $\omega^8 = 1$.

$$\omega^8 = \left(\frac{1}{\sqrt{2}}\right)^8 (1+i)^8$$

$$= \frac{1}{16}(1+i)^8$$

$$= \frac{1}{16}\sum_{j=0}^{8}\binom{8}{j} i^j$$

I had to change my usual subscript of i to j to avoid confusion!

$$= \frac{1}{16}\left[\binom{8}{0} i^0 + \binom{8}{1} i^1 + \binom{8}{2} i^2 + \binom{8}{3} i^3 + \binom{8}{4} i^4 + \binom{8}{5} i^5 + \binom{8}{6} i^6 + \binom{8}{7} i^7 + \binom{8}{8} i^8\right]$$

$$= \frac{1}{16}\left[1 + \cancel{8i} + \overset{-28}{\cancel{28i^2}} + \overset{\cancel{56i}}{\cancel{56i^3}} + 70\cancel{i^4} + \underset{\cancel{56i}}{\cancel{56i^5}} + \underset{-28}{\cancel{28}i^6} + \underset{\cancel{8i}}{\cancel{8i^7}} + \underset{1}{\cancel{i^8}}\right]$$

$$= \frac{1}{16}\left[1 + -28 - 28 + 1 + 70\right] = \underline{\underline{1}}$$

4. We use De Moivre's theorem and the binomial theorem. By De Moivre's theorem

$$(\cos\theta + i\sin\theta)^{10} = \cos 10\theta + i\sin 10\theta. \quad (*)$$

By the binomial theorem

$$(\cos\theta + i\sin\theta)^{10} = \sum_{j=0}^{10} \binom{10}{j} (\cos\theta)^{10-j} (i\sin\theta)^{j} \quad \boxed{+}$$

We need to calculate powers of i:

i^0	i	i^2	i^3	i^4	i^5	i^6	i^7	i^8	i^9	i^{10}
1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1

real powers of i

Now equate real and imaginary parts of $(*)$ and $\boxed{+}$

$j = 0, 2, 4, 6, 8, 10$

$\cos 10\theta =$

$$\binom{10}{0}\cos^{10}\theta + \binom{10}{2}\cos^{8}\theta\,(i\sin\theta)^{2}$$

$$+ \binom{10}{4}\cos^{6}\theta\,(i\sin\theta)^{4} + \binom{10}{6}\cos^{4}\theta\,(i\sin\theta)^{6}$$

$$+ \binom{10}{8}\cos^{2}\theta\,(i\sin\theta)^{8} + \binom{10}{10}(i\sin\theta)^{10}$$

$$= \binom{10}{0}\cos^{10}\theta - \binom{10}{2}\cos^{8}\theta\sin^{2}\theta$$

$$+ \binom{10}{4}\cos^{6}\theta\sin^{4}\theta - \binom{10}{6}\cos^{4}\theta\sin^{6}\theta$$

$$+ \binom{10}{8}\cos^{2}\theta\sin^{8}\theta - \binom{10}{10}\sin^{10}\theta$$

7(a)

$\binom{10}{0} = 1$, $\binom{10}{10} = 1$, $\binom{10}{4} = 210$

$\binom{10}{2} = 45$, $\binom{10}{8} = 45$, $\binom{10}{6} = 210$

$\therefore$

$\cos 10\theta =$

$\cos^{10}\theta - 45\cos^8\theta\sin^2\theta + 210\cos^6\theta\sin^4\theta$

$- 210\cos^4\theta\sin^6\theta + 45\cos^2\theta\sin^8\theta - \sin^{10}\theta$

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5. By the theory in the lecture, any integral roots must divide -24. We quickly find that $1, 2$, and 3 are roots. By dividing out by the linear factors $(x-1)$, $(x-2)$ and $(x-3)$ we find that

$$p(x) = (x-1)(x-2)(x-3)(x^4+5x^2+4).$$

We are given that i is a root of $p(x)$ and so i is a root of x^4+5x^2+4.

By the theory, $-i$ must also be a root.

So, $(x-i)(x+i) = x^2+1$ must divide x^4+5x^2+4.

In fact,

$$x^4+5x^2+4 = (x^2+1)(x^2+4).$$

There are two ways to show this:

(1) Long division (see my book)

(2) We need to find $x^2 + ax + b$

such that

$$x^4 + 5x^2 + 4 = (x^2+1)(x^2+ax+b)$$

$$= x^4 + ax^3 + bx^2 + x^2 + ax + b$$

$$= x^4 + ax^3 + (b+1)x^2 + ax + b$$

We see that $a = 0$, $b = 4$ works.

Thus $x^4 + 5x^2 + 4 = (x^2+1)(x^2+4)$,

as claimed.

Check that x^2+1 and x^2+4 are irreducible (discriminants < 0)

Thus

$$\boxed{p(x) = (x-1)(x-2)(x-3)(x^2+1)(x^2+4)}$$

Is the decomposition of $p(x)$ as a product of real linear polynomials and real irreducible quadratic polynomials.