Lecture 18

Section 3: Matrices

Definition A matrix (pl. matrices) is a rectangular array of numbers (usually but other mathematical quantities can also occur).

$$\begin{array}{cc} & \text{columns} \\ \text{rows} & \begin{pmatrix} 3 & 4 & 6 & 7 \\ 8 & 0 & -\pi & e \\ \sqrt{2} & 1 & i & -i \end{pmatrix} \end{array} \qquad 3\times 4 \text{ Matrix}$$

In general, an $m \times n$ matrix has m rows and n columns. $m \times n$ is called the size of the matrix. We usually denote matrices by upper case Latin letters: $A, B, C, \ldots$

$(A)_{ij}$ or A_{ij} is the element of A in row i and column j.

<u>Example</u>

Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$

This is a 2×4 matrix. $(A)_{23} = 7$.

<u>Definition</u> Matrices A and B are <u>equal</u> precisely when

(1) They have the same size.

(2) Corresponding elements/entries are the same i.e. $(A)_{ij} = (B)_{ij}$ for all appropriate values of i and j.

<u>Scalars</u> In matrix theory, numbers tend to be referred to as <u>scalars</u> and denoted by l.c. Greek letters: $\lambda, \mu, \ldots$

Matrix arithmetic

Addition Matrices A and B can only be added if they have the same size. If they don't then addition is not defined.

Then

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij}$$

Example $\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0+1 & 1+2 \\ -1+2 & 2+1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$$

Subtraction If A and B have the same size

then $(A-B)_{ij} = (A)_{ij} - (B)_{ij}$.

Scalar multiplication

$$(\lambda A)_{ij} = \lambda (A)_{ij}.$$

Example $\quad 2\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}.$

The transpose of A

$$(A^T)_{ij} = (A)_{ji}$$

Example $\quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}^T = \begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{pmatrix}$

multiplication

We need some ~~[illegible]~~ preparation.

$(a_1\ a_2 \dots a_m)$ is called a

row vector or row matrix.

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$ is called a Column vector or Column matrix

The inner product of a row matrix and a column matrix

Let $\underline{a} = (a_1 \dots a_m)$ and

$\underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$. Then $\underline{a} \cdot \underline{b}$ is defined

inner product

only if $m = n$ in which case

$$\underline{a} \cdot \underline{b} = a_1 b_1 + \dots + a_m b_m$$

$$= \sum_{i=1}^{m} a_i b_i \quad , \quad \text{a single number.}$$

<u>Example</u> if $\underline{a} = \cancel{\begin{pmatrix}1\\2\\3\end{pmatrix}} \begin{pmatrix}1 & 2 & 3\end{pmatrix}$

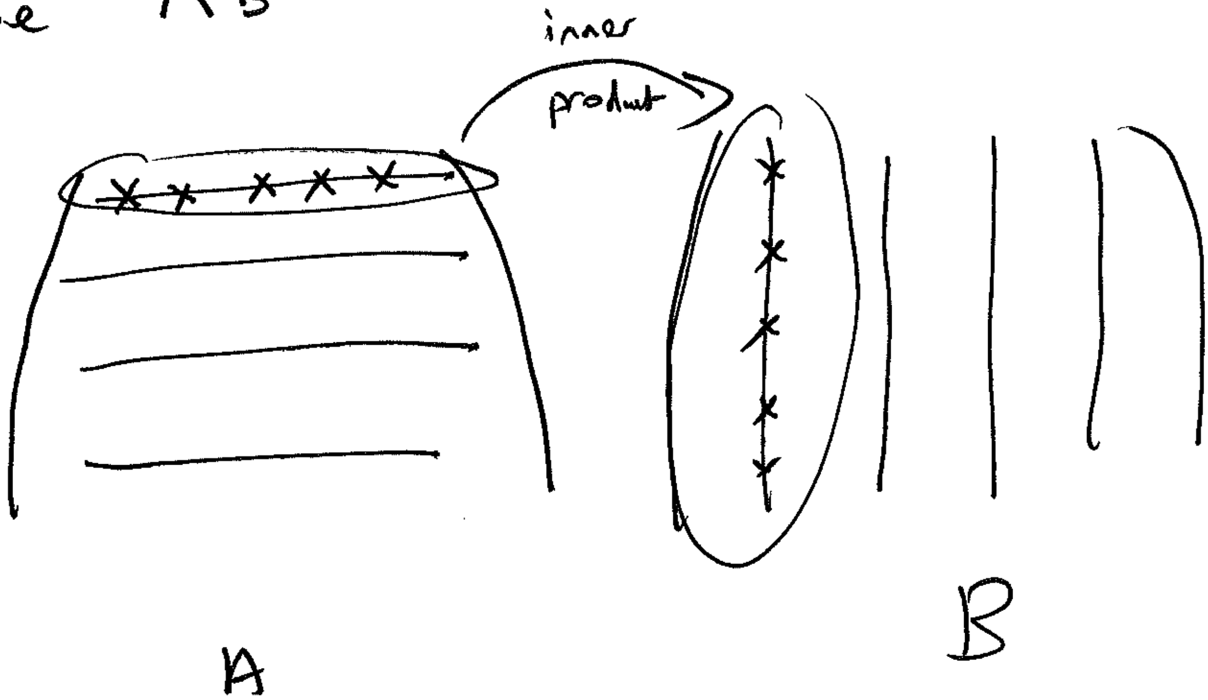
and $\underline{b} = \begin{pmatrix}0\\-1\\2\end{pmatrix}$ Then

$$\underline{a} \cdot \underline{b} = 1 \times 0 + (-1) \times 2 + 2 \times 3$$

$$= -2 + 6 = \underline{\underline{4}}$$

Matrix multiplication

This is more complex than the other matrix operations, but very important. Want to define AB



Think of A as being a sequence of row matrices

Think of B as being a sequence of column matrices

Need for AB to be defined:

$\#$ columns of A = $\#$ rows of B
(else not defined)

$$\underset{\substack{\shortparallel \\ A}}{\begin{pmatrix} \longleftarrow & \underline{a}_1 & \longrightarrow \\ & \vdots & \\ \longleftarrow & \underline{a}_m & \longrightarrow \end{pmatrix}} \underset{\substack{\shortparallel \\ B}}{\begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \underline{b}_1 & \underline{b}_2 & \cdots & \underline{b}_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}}$$

$$= \underset{\substack{\shortparallel \\ AB}}{\begin{pmatrix} \underline{a}_1 \cdot \underline{b}_1 & \underline{a}_1 \cdot \underline{b}_2 & \cdots & \underline{a}_1 \cdot \underline{b}_n \\ & \vdots & & \\ \underline{a}_m \cdot \underline{b}_1 & \underline{a}_m \cdot \underline{b}_2 & \cdots & \underline{a}_m \cdot \underline{b}_n \end{pmatrix}}$$

Example

$$2 \times \overbrace{3 \cdot 3}^{=} \times 2$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (1\ 2\ 1) \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} & (1\ 2\ 1) \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \\ (0\ 1\ -1) \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} & (0\ 1\ -1) \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1+2+0 & 2-2+1 \\ 0+1+0 & 0-1-1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}$$

If A is $m \times p$ and B is $p \times n$

then AB is $m \times n$.

As usual, $A^2 = AA$

$$A^3 = (AA)A$$

if defined.