Lecture 19

Matrix algebra

Matrix addition

(MA1) $\quad (A+B)+C = (A+B)+C$

Associativity.

(MA2) $\quad A + O = A = O + A$.

Here O is the zero matrix (of the appropriate size)

Additive identity.

(MA3) $\quad A + (-A) = O = (-A) + A$

where $A = -1A$, the additive inverse of A.

(MA4) $\quad A + B = B + A$

Matrix addition is commutative.

Properties of scalar multiplication

(S1) $1A = A, \quad -1A = -A.$

(S2) $0A = 0$

(S3) $\lambda(A+B) = \lambda A + \lambda B.$ $\quad$ λ, μ scalars

(S4) $(\lambda\mu)A = \lambda(\mu A).$

(S5) $(\lambda+\mu)A = \lambda A + \mu A.$

(S6) $(\lambda A)B = A(\lambda B) = \lambda(AB)$

Properties of the transpose

(T1) $(A^T)^T = A$.

(T2) $(A+B)^T = A^T + B^T$.

(T3) $(\lambda A)^T = \lambda A^T$

(T4) $(AB)^T = B^T A^T$ $\leftarrow$ NB

Counterexamples

(1) Matrix multiplication is not commutative.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -1 & 7 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 2 & 2 \end{pmatrix}$$

(2) Two non-zero matrices can multiply to zero

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} -2 & -6 \\ 1 & 3 \end{pmatrix}$$

$$A.B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -2 & -6 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(3) <u>Cancellation of Matrices is not allowed</u>

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 1 \\ 1 & 4 \end{pmatrix}$$

$$AB = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 8 \\ 1 & 4 \end{pmatrix}$$

$$AC = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 8 \\ 1 & 4 \end{pmatrix}$$

$\therefore \; AB = AC$

But $B \neq C$.

Properties of matrix multiplication

(MM1) $$(AB)C = A(BC)$$

(MM2) $$AI = A = IA$$

where I is the identity matrix of the appropriate size

$$I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \quad \text{square.}$$

(MM2) $$AO = O = OA$$

where O is the zero matrix of the appropriate size

(MM3) $$A(B+C) = AB + AC$$

$$(B+C)A = BA + CA$$

All of these properties can be proved from the definitions.

Associativity of Matrix multiplication

We prove that $(AB)C = A(BC)$.

(1) Need to show that $(AB)C$ and $A(BC)$ have the same size.

size of $(AB)C$ is $m \times q$

$$\begin{array}{ccc} A & B & C \\ m \times p & p \times n & n \times q \end{array}$$

size of $A(BC)$ is $m \times q$

(2) We now prove that

$$((AB)C)_{ij} = (A(BC))_{ij}$$

for all appropriate values of i and j

Convention

$$(X)_{ij} = x_{ij}$$

$$(XY)_{ij} = \sum_{k} x_{ik}\, y_{kj}$$

$$((AB)\,C)_{ij} = \sum_{k} (AB)_{ik}\, c_{kj}$$

$$= \sum_{k} \left(\sum_{l} a_{il}\, a_{lk} \right) c_{kj}$$

$$= \sum_{k} \sum_{l} a_{il}\, a_{lk}\, a_{kj} \quad (*)$$ using distributivity of scalars.

$$\left(A\,(BC)\right)_{ij} = \sum_{l} a_{il}\,(BC)_{lj}$$

$$= \sum_{l} a_{il} \left(\sum_{k} b_{lk}\, c_{kj} \right)$$

$$= \sum_{l} \sum_{k} a_{il}\, b_{lk}\, c_{kj} \quad (+) \text{ distr. of scalars.}$$

(*) and (+) are the same – just added up in a different order.