Lecture 20

– proof of associativity of matrix multiplication (concluded from last time).

Linear equations

__Example__ Consider the following system of three linear equations in four unknowns:

This explains the form of matrix multiplication

$$\begin{aligned} 3x - 2y + z + 6t &= 5 \\ x + 3y - z + 5t &= 4 \\ -2x - y - z - 2t &= 3 \end{aligned}$$

This can be written in matrix form

$$\underbrace{\begin{pmatrix} 3 & -2 & 1 & 6 \\ 1 & 3 & -1 & 5 \\ -2 & -1 & -1 & -2 \end{pmatrix}}_{\text{matrix of coefficients}} \underbrace{\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}}_{\text{unknown}} = \underbrace{\begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}}_{\text{given}}$$

This has the form $$A\underline{x} = \underline{b} \quad (*)$$

A system of linear equations

where $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, A: $m \times n$ and $\underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

Any column vector $\underline{a}$ s.t. $A\underline{a} = \underline{b}$ is called a solution of $(*)$

The solution set of $A\underline{x} = \underline{b}$ is the set

$$\{\underline{a} : A\underline{a} = \underline{b}\}$$

Theorem (Fundamental theorem of linear equations)

Let $A\underline{x} = \underline{b}$ be a system of linear equations (over $\mathbb{C}$). ~~Then~~ Then there are 3 possibilities:

(1) There are no solutions } inconsistent

(2) Exactly one solution

(3) Infinitely many solutions } consistent

Proof Assume that there are at least two distinct solutions $\underline{u}$ and $\underline{v}$ to the system $A\underline{x} = \underline{b}$.

So, $A\underline{u} = \underline{b}$ and $A\underline{v} = \underline{b}$.

Then $A(\underline{u}-\underline{v}) = A\underline{u} - A\underline{v} = \underline{b} - \underline{b} = \underline{0}$

Let λ be any scalar ($\lambda \in \mathbb{C}$).

$$A\left(\lambda(\underline{u}-\underline{v})+\underline{u}\right) = \lambda A(\underline{u}-\underline{v}) + A\underline{u}$$
$$= \underline{0}+\underline{b} = \underline{b}$$

$\therefore$ For $\underline{\underline{\text{all}}}$ $\lambda \in \mathbb{C}$,

$\lambda(\underline{u}-\underline{v})+\underline{u}$ are solutions

We have therefore found ∞ly many solutions $\blacksquare$

System of linear equations

- inconsistent?
- Consistent?
  - Unique solution?
  - infinitely many solutions?

We shall develop an algorithm to answer these questions.