Lecture 21

Algorithm for solving linear equations

Elementary row operations (EROs)

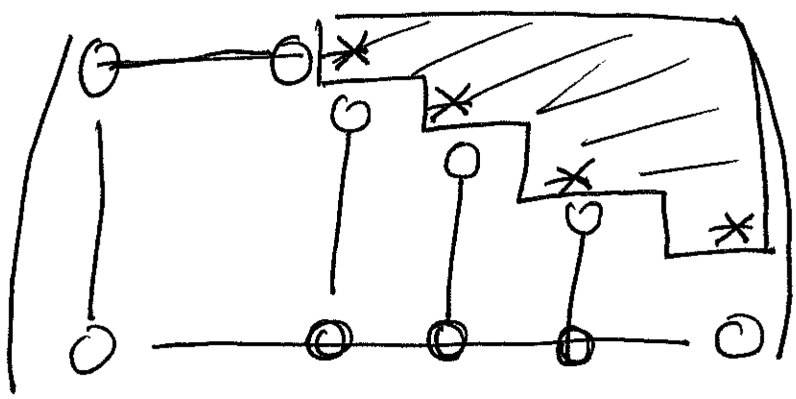
1. $R_i \leftarrow \lambda R_i$ (where $\lambda \neq 0$).
Multiply row i by a non-zero scalar λ.

2. $R_i \leftrightarrow R_j$. Interchange row i and row j.

3. $R_j \leftarrow R_j + \lambda R_i$. Add λ times row i to row j

Proposition Let $A\underline{x} = \underline{b}$. Apply any finite number of EROs to the augmented matrix $(A \mid \underline{b})$ to get $(A' \mid \underline{b}')$. Then the solution set of $A\underline{x} = \underline{b}$ = the solution set of $A'\underline{x} = \underline{b}'$.

The matrix $(A' | \underline{b}')$ is an **echelon matrix** if it looks like this:



$*$ means $\neq 0$

staircase pattern = echelon

The algorithm for solving a system of linear equations works as follows.

1. Input: $A\underline{x} = \underline{b}$.
2. Write as augmented matrix $(A | \underline{b})$.
3. Convert to echelon matrix $(A' | \underline{b}')$ using EROs.
4. Solve $A'\underline{x} = \underline{b}'$ (now easy because of (3)).

Special case when A is 3×3

$$\left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right)$$

use top left non-zero entry to produce 0s below it using EROs.

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{array}\right)$$

Now repeat with leading non-zero entry in next row.

Various patterns can arise:

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{array}\right) \longleftarrow \text{inconsistent}$$

non-zero!

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right) \longleftarrow \text{consistent, unique solution}$$

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & 0 \end{array}\right) \longleftarrow \text{consistent, infinitely many solutions}$$

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$