Lecture 22

Examples

1. We show that the following system is inconsistent (= no solution).

$$\left.\begin{aligned} x + 2y - 3z &= -1 \\ 3x - y + 2z &= 7 \\ 5x + 3y - 4z &= 2 \end{aligned}\right\}$$ system of linear equations

Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{array}\right)$$

We now apply the algorithm (often called Gaussian elimination):

Augmented matrices	Calculations
$\left(\begin{array}{ccc\|c} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{array}\right)$	
$\left(\begin{array}{ccc\|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 5 & 3 & -4 & 2 \end{array}\right)$	$R_2 \leftarrow R_2 - 3R_1$ $\begin{array}{ccc\|c} 3 & -1 & 2 & 7 \\ -3 & -6 & 9 & 3 \\ \hline 0 & -7 & 11 & 10 \end{array}$
$\left(\begin{array}{ccc\|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & -7 & 11 & 7 \end{array}\right)$	$R_3 \leftarrow R_3 - 5R_1$ $\begin{array}{ccc\|c} 5 & 3 & -4 & 2 \\ -5 & -10 & 15 & 5 \\ \hline 0 & -7 & 11 & 7 \end{array}$
$\left(\begin{array}{ccc\|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ \underbrace{0 \quad 0 \quad 0}_{\text{all 0s}} & & & \underbrace{-3}_{\text{non-zero}} \end{array}\right)$	$R_3 \leftarrow R_3 - R_2$ $\begin{array}{ccc\|c} 0 & -7 & 11 & 7 \\ 0 & 7 & -11 & -10 \\ \hline 0 & 0 & 0 & -3 \end{array}$

This is an echelon matrix.

Last row corresponds to the equation

$0x + 0y + 0z = -3$ i.e. $0 = -3$.

This is non-sense, so system is <u>inconsistent</u>.

2. We show that the following system has exactly one solution.

$$\left.\begin{aligned} x + 2y + 3z &= 4 \\ 2x + 2y + 4z &= 0 \\ 3x + 4y + 5z &= 2. \end{aligned}\right\} \text{ System of linear equations } (*)$$

Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 0 \\ 3 & 4 & 5 & 2 \end{array}\right).$$

Augmented matrices	Calculations
$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 0 \\ 3 & 4 & 5 & 2 \end{array}\right)$	
$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -8 \\ 3 & 4 & 5 & 2 \end{array}\right)$	$R_2 \leftarrow R_2 - 2R_1$ $\begin{array}{ccc\|c} 2 & 2 & 4 & 0 \\ -2 & -4 & -6 & -8 \\ \hline 0 & -2 & -2 & -8 \end{array}$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -8 \\ 0 & -2 & -4 & -10 \end{array}\right)$$

$R_3 \leftarrow R_3 - 3R_1$

$$\begin{array}{ccc|c} 3 & 4 & 5 & 2 \\ -3 & -6 & -9 & -12 \\ \hline 0 & -2 & -4 & -10 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -8 \\ 0 & 0 & -2 & -2 \end{array}\right)$$

Echelon Matrix

$R_3 \leftarrow R_3 - R_2$

$$\begin{array}{ccc|c} 0 & -2 & -4 & -10 \\ 0 & 2 & 2 & 8 \\ \hline 0 & 0 & -2 & -2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

"Tidied up"

$R_2 \leftarrow -\frac{1}{2} R_2$

$R_3 \leftarrow -\frac{1}{2} R_3$

The corresponding system of equations is:

$$\begin{aligned} x + 2y + 3z &= 4 \\ y + z &= 4 \\ \boxed{z = 1} \end{aligned}$$

back substitution

$$y = 4 - z = 4 - 1 = 3$$

$$x = 4 - 2y - 3z = 4 - 2.3 - 3.1$$

$$= 4 - 6 - 3 = 4 - 9 = -5$$

We claim that the unique solution to our original system of equations $(*)$ is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix}.$$

<u>We now CHECK that we really have solved the original system of equations because that is what we were asked:</u>

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -5+6+3 = 4 \\ -10+6+4 = 0 \\ -15+12+5 = 2 \end{pmatrix}$$

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<u>YOU MUST CHECK ALL ORIGINAL EQUATIONS</u>

3. We show that the followings system has infinitely many solutions:

$$\left.\begin{array}{l} x + 2y - 3z = 6 \\ 2x - y + 4z = 2 \\ 4x + 3y - 2z = 14 \end{array}\right\}$$ system of linear equations ($\square$).

Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 2 & -1 & 4 & 2 \\ 4 & 3 & -2 & 14 \end{array}\right)$$

Carry out the following EROs !-

- $R_2 \leftarrow R_2 - 2R_1$; $R_3 \leftarrow R_3 - 4R_1$ to get

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & -5 & 10 & -10 \\ 0 & -5 & 10 & -10 \end{array}\right)$$

Now carry out $R_3 \leftarrow R_3 - R_2$ to get

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & -5 & 10 & -10 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Apply $R_2 \leftarrow \frac{1}{-5} R_2$ to tidy up.

Whole row is zero, and so infinitely many solutions

[redacted].

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$ echelon matrix

The corresponding system of linear equations is

$$x + 2y - 3z = 6$$
$$y - 2z = 2$$

called the free variable

z can take any value. Put $z = \lambda$, a scalar.

Then $y = 2 + 2z = 2 + 2\lambda$

and

$$x = 6 - 2y + 3z$$
$$= 6 - 2(2 + 2\lambda) + 3\lambda$$
$$= 6 - 4 - 4\lambda + 3\lambda$$
$$= 2 - \lambda$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-\lambda \\ 2+2\lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

either is acceptable

where $\lambda \in \mathbb{R}$ is <u>any</u> real number.

We see that there are infinitely many solutions. We must show that they <u>all</u> satisfy our original equations $\Box$:

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} 2-\lambda \\ 2+2\lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} 2 \cancel{-\lambda} + 4 \cancel{+4\lambda} \cancel{-3\lambda} \\ 4 \cancel{-2\lambda} - 2 \cancel{-2\lambda} \cancel{+4\lambda} \\ 8 \cancel{-4\lambda} + 6 \cancel{+6\lambda} \cancel{-2\lambda} \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 2 \\ 14 \end{pmatrix}$$

4. <u>A system of linear equations with 2 free variables</u>.

$$x + 2y + 3z = 4$$

Put $z = \lambda \ (\in \mathbb{R})$ and $y = \mu \ (\in \mathbb{R})$.

Then $x = 4 - 2y - 3z = 4 - 2\mu - 3\lambda$.

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 - 2\mu - 3\lambda \\ \mu \\ \lambda \end{pmatrix} \text{ where } \lambda, \mu \in \mathbb{R}.$$