Lecture 23

Definition A square, $n \times n$ matrix, A is said to be **invertible** if there is an $n \times n$ matrix B s.t. $AB = I = BA$ where I is the $n \times n$ identity matrix. We say that B is **an inverse of A**.

Lemma Let B and C be inverses of A. Then $B = C$.

Proof By defn, $BA = I = AB$ and $CA = I = AC$.

From $BA = I$ we get $(BA)C = IC = C$

But $(BA)C = B(AC) = BI = B$. $\therefore B = C$. ■

It follows by the above lemma that IF A HAS AN INVERSE THAT INVERSE IS UNIQUE; in this case, we write A^{-1} for the unique inverse of A.

Two important questions:

(1) How do we decide if A is invertible or not?

(2) If A is invertible how do we compute A^{-1}?

The answers to both of these questions use determinants.

The determinant

Definition Let A be a square matrix.

We define the determinant of A, written $\det(A)$

or $\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{n1} & \dots & & a_{nn} \end{vmatrix}$, as follows.

$$\det(a) = a$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \underline{\underline{a}}\begin{vmatrix} e & f \\ h & i \end{vmatrix} - \underline{\underline{b}}\begin{vmatrix} d & f \\ g & i \end{vmatrix}$$

$$+ \underline{\underline{c}}\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

; Pay attention to signs!

Exaple

(1) $$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 2 \times 5 - 3 \times 4$$
$$= 10 - 12 = \underline{\underline{-2}}$$

(2) $$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix}$$
$$+ 1 \begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix}$$

$$= 1 - 2(3) + (-2)$$
$$= 1 - 6 - 2 = \underline{\underline{-7}}$$

The key property of dets we shall need in this section is the following. I shall not prove it in generality but the proof of the 2×2 case can be found in my book pp 249-250.

<u>Theorem</u> Let A and B be $n \times n$ matrices.
Then $\det(AB) = \det(A)\det(B)$.

<u>Lemma</u> If A is invertible then $\det(A) \neq 0$.

<u>Proof</u> By definition, if A is invertible then $AA^{-1} = I$.
Thus $\det(AA^{-1}) = \det(I) = 1$.
But, by the theorem, $\det(AA^{-1}) = \det(A)\det(A^{-1})$.
It follows that $\det(A) \neq 0$. ■

In fact,

$$\boxed{A \text{ is invertible} \iff \det(A) \neq 0.}$$

We shall prove this result in the 2×2 and 3×3 cases.

Let A be an $n \times n$ matrix. The <u>adjugate of A</u>, written $\operatorname{adj}(A)$, is a matrix s.t.

$$A \operatorname{adj}(A) = \det(A)\, I = \operatorname{adj}(A)\, A.$$

Observe that if $\det(A) \neq 0$ then $\boxed{A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)}$.

The adjugate of an arbitrary 2x2 matrix

Input $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Procedure:

- For each entry, cross out the row and column containing that entry and write down the number you see:

$$\begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

- Add signs $(-1)^{i+j}$ where i is the row number and j is the column number:

$$\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

> - NUMBER
> - SIGNS
> - TRANSPOSE

- Take the transpose

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \mathrm{adj}(A)$$

<u>Check</u>

$$A \, \mathrm{adj}(A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \begin{pmatrix} ad-bc & -ab+ba \\ cd-dc & -bc+da \end{pmatrix}$$

$$= \det(A) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It follows that if $\det(A) \neq 0$ then

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$