Lecture 24

3 × 3 case

This is identical to the 2 × 2 case except that in the first step, for each entry, cross out its row and column and write down the determinant of the matrix you see. The remaining steps are the same.

$$\boxed{\begin{array}{l} (1)\ \text{Number} \\ (2)\ \text{Sign} \\ (3)\ \text{Transpose} \end{array}}$$

Example We compute the adjugate of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

- $$\begin{pmatrix} \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -1 & 5 & 2 \\ 1 & 5 & 3 \\ 2 & -5 & -4 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

- Now multiply entry in row i and column j by $(-1)^{i+j}$.

$$\begin{pmatrix} -1 & -5 & 2 \\ -1 & 5 & -3 \\ 2 & 5 & -4 \end{pmatrix}$$

- Now take the transpose

$$\begin{pmatrix} -1 & -1 & 2 \\ -5 & 5 & 5 \\ 2 & -3 & -4 \end{pmatrix} = \operatorname{adj}(A)$$

Check

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} -1 & -1 & 2 \\ -5 & 5 & 5 \\ 2 & -3 & -4 \end{pmatrix}$$

$$= -5 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It follows that

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ -1 & 1 & 2 \end{pmatrix}^{-1} = \frac{-1}{5} \begin{pmatrix} -1 & -1 & 2 \\ -5 & 5 & 5 \\ 2 & -3 & -4 \end{pmatrix}$$

Calculation

$$
\underset{\text{matrix}}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ -1 & 1 & 2 \end{pmatrix}}
\underset{\text{adjugate}}{\begin{pmatrix} -1 & -1 & 2 \\ -5 & 5 & 5 \\ 2 & -3 & -4 \end{pmatrix}} =
$$

$$
\begin{pmatrix}
\overset{=-5}{-1-10+6} & \overset{,,\ 0}{-1+10-9} & \overset{,,\ 0}{2+10-12} \\
-2+2=0 & \overset{=-5}{-2-3} & 4-4=0 \\
1-5+4=0 & 1+5-6=0 & \overset{=-5}{-2+5-8}
\end{pmatrix}
$$

$$
= \begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{pmatrix} = -5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
= -5I
$$

The characteristic polynomial and eigenvalues

Definition Let A be a square matrix.

Define $\chi_A(x) = \det(A - xI)$

is a polynomial called the characteristic polynomial of A.

The roots of $\chi_A(x)$ are called the eigenvalues of A.

Example

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Then

$$\chi_A(x) = \det(A - xI)$$

$$= \det\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \right)$$

$$= \det \begin{pmatrix} 1-x & 2 \\ 3 & 4-x \end{pmatrix} = (1-x)(4-x) - 6$$

$$= 4 - x - 4x + x^2 - 6$$

$$= \underline{\underline{x^2 - 5x - 2}}$$

The eigenvalues of A are the roots of $x^2 - 5x - 2$

these are:

$$\frac{5 \pm \sqrt{25 + 8}}{2} = \frac{5 \pm \sqrt{33}}{2}$$

Theorem (Cayley-Hamilton)

Each square matrix is a root of its characteristic polynomial.

Example We have seen that the characteristic polynomial of $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is $x^2 - 5x - 2$.

$(= x^2 - 5x - 2x^0)$

NB

We calculate

$$A^2 - 5A - \underline{\underline{2I}}$$

$$A^2 - 5A - 2I = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^2 - 5\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} + \begin{pmatrix} -5 & -10 \\ -15 & -20 \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}}$$

(As claimed by the Cayley-Hamilton theorem.)

$$\boxed{\text{END OF SECTION 3}}$$

Aside

A matrix A is called <u>symmetric</u> if $A = A^T$.

It is a remarkable theorem that the roots of the characteristic polynomial of a symmetric matrix are always <u>real</u>.

Revision of Section 3

1. Let $A = \begin{pmatrix} 2 & 2 & -5 \\ 3 & 7 & -15 \\ 1 & 2 & -4 \end{pmatrix}$.

(i) Calculate $A^3 - 5A^2 + 7A - 3I$.

(ii) Calculate the determinant of A.

(iii) Calculate the inverse of A and show that your solution works.

(iv) Calculate the characteristic polynomial of A and the eigenvalues of A along with their multiplicities.

2. Solve the following system of equations and show that your solutions work:

$$\begin{aligned} x + 2y + z &= 0 \\ 2x - y + 3z &= 0 \\ 3x + y + 4z &= 0. \end{aligned}$$

Solutions to revision exercises

1

(i) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(ii) 3

(iii) The inverse is

$$\frac{1}{3}\begin{pmatrix} 2 & -2 & 5 \\ -3 & -3 & 15 \\ -1 & -2 & 8 \end{pmatrix}$$

(iv) The characteristic polynomial is $(x-1)^2(x-3)$
$= x^3 - 5x^2 + 7x - 3$. The eigenvalues are 1 (twice) and 3 (once).

2. The solution space is

$$\left\{ \lambda \begin{pmatrix} -7/5 \\ 1/5 \\ 1 \end{pmatrix} : \lambda \in \mathbb{R} \right\}$$