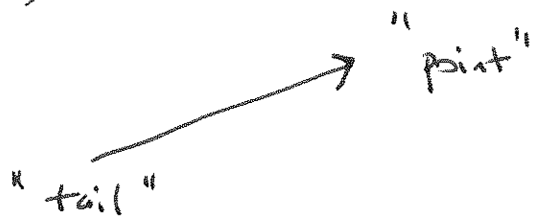
Lecture 25

4. Vectors

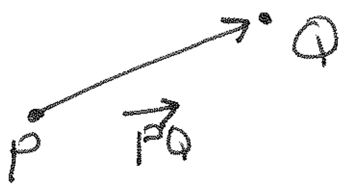
We begin by describing (free) vectors geometrically. Afterwards, we shall describe them algebraically.

Definition A (free) vector is a directed line segment (in space) which can be moved parallel to itself.



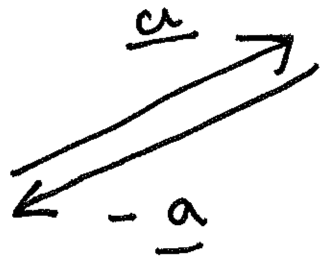
Vectors are denoted by bold letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$

If P and Q are points, then $\overrightarrow{PQ}$ is the vector determined by the directed line segment from P to Q.



The directed line segment determined by P and P is just a point. This determines the zero vector $\mathbf{0}$

If $\underline{a}$ is a vector, then $-\underline{a}$ is the vector pointing in the opposite direction.



If $\underline{a}$ and $\underline{b}$ are vectors then we may add them - this may require $\underline{a}$ (or $\underline{b}$) to be moved so that the point of $\underline{a}$ touches the tail of $\underline{b}$:

<u>Properties of vector addition</u>

(VA1) $\underline{a} + (\underline{b} + \underline{c}) = (\underline{a} + \underline{b}) + \underline{c}$

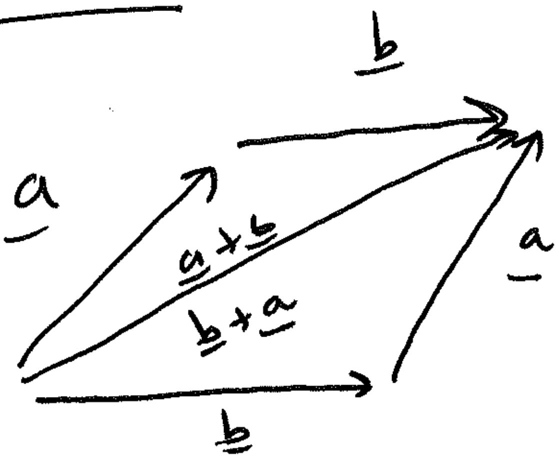
(VA2) $\underline{0} + \underline{a} = \underline{a} + \underline{0} = \underline{a}$.

(VA3) $\underline{a} + (-\underline{a}) = \underline{0} = (-\underline{a}) + \underline{a}$.

(VA4) $\underline{a} + \underline{b} = \underline{b} + \underline{a}$

The proofs of these results are carried out geometrically.

Example $\underline{a}+\underline{b}=\underline{b}+\underline{a}$



Let $\underline{a}$ be a vector.

Define $\|\underline{a}\|$, the length of a.

$\|\underline{a}\| \geqslant 0$ and $\|\underline{a}\| = 0 \iff \underline{a} = \underline{0}$

Can multiply a vector $\underline{a}$ by a scalar $\lambda \ (\in \mathbb{R})$

to get a vector $\lambda \underline{a}$

- If $\lambda > 0$ then $\lambda \underline{a}$ has the same direction as $\underline{a}$ but length $\lambda \|\underline{a}\|$.

• If $\lambda = 0$ then $\lambda \underline{a} = \underline{0}$, the zero vector

• If $\lambda < 0$ then $\lambda \underline{a}$ has the direction $-\underline{a}$ and length $|\lambda| \, \|\underline{a}\|$

If $\underline{a} \neq \underline{0}$ then $\hat{\underline{a}} = \frac{1}{\|\underline{a}\|} \underline{a}$ is a vector of unit length called a <u>unit vector</u> pointing in the same direction as $\underline{a}$

<u>Properties of scalar multiplication</u>

$\lambda, \mu \in \mathbb{R}$

(SM1). $0 \, \underline{a} = \underline{0}$.

(SM2). $1 \, \underline{a} = \underline{a}$.

(SM3). $(-1) \, \underline{a} = -\underline{a}$.

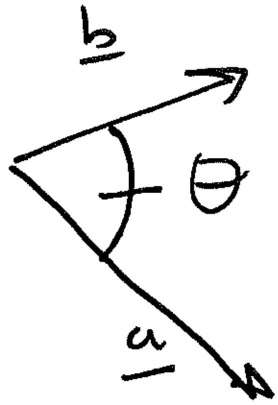
(SM4). $(\lambda + \mu) \underline{a} = \lambda \underline{a} + \mu \underline{a}$.

(SM5). $\lambda (\underline{a} + \underline{b}) = \lambda \underline{a} + \lambda \underline{b}$.

(SM6) $\lambda (\mu \underline{a}) = (\lambda \mu) \underline{a}$

Operations on vectors

If $\underline{a}, \underline{b} \neq \underline{0}$ denote the angle ~~between~~ them by θ (where $0 \leq \theta \leq \pi$)



<u>Definition</u> $\underline{a} \cdot \underline{b} = \|\underline{a}\| \|\underline{b}\| \cos\theta$

If $\underline{a}$ or $\underline{b}$ is $\underline{0}$ then define $\underline{a} \cdot \underline{b} = 0$

This is called the <u>inner product of $\underline{a}$ and $\underline{b}$</u>

Properties of the inner product

All easy to prove

(IP1) $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$.

(IP2) $\underline{a} \cdot \underline{a} = \|\underline{a}\|^2$.

(IP3) $\lambda(\underline{a} \cdot \underline{b}) = (\lambda \underline{a}) \cdot \underline{b} = \underline{a} \cdot (\lambda \underline{b})$

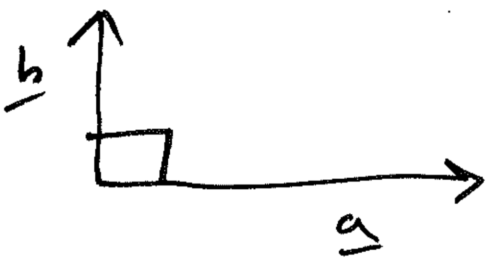
where $\lambda \in \mathbb{R}$.

See my book for a proof

$\boxed{\text{(IP4)} \quad \underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}}$

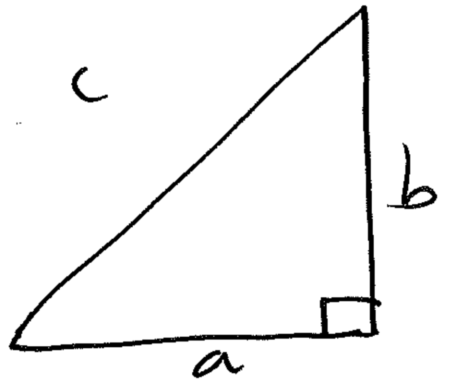
If $\underline{a}, \underline{b} \neq \underline{0}$ and $\underline{a} \cdot \underline{b} = 0$

we say that $\underline{a}$ and $\underline{b}$ are <u>orthogonal</u>



Application of the inner product:

Proof of Pythagoras' theorem



$$a^2 + b^2 = c^2$$

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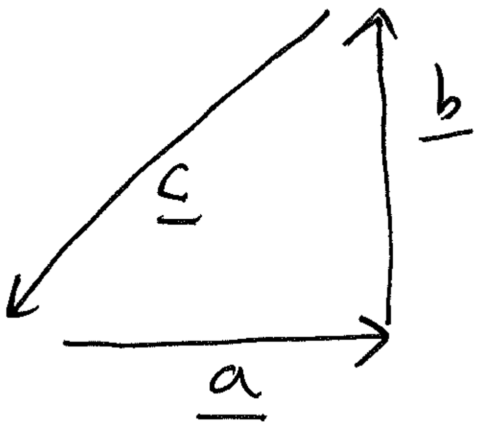
let



where $\|\underline{a}\| = a$

$\|\underline{b}\| = b$

$\|\underline{c}\| = c.$

Observe that $\underline{a} + \underline{b} + \underline{c} = \underline{0}$.

Thus $\underline{a} + \underline{b} = -\underline{c}$.

Take inner products of both sides

$$(\underline{a} + \underline{b}) \cdot (\underline{a} + \underline{b}) = (-\underline{c}) \cdot (-\underline{c})$$
$$= \underline{c} \cdot \underline{c} = \|\underline{c}\|^2$$

But by (IP4)

$$(\underline{a} + \underline{b}) \cdot (\underline{a} + \underline{b}) = \underline{a} \cdot \underline{a} + \underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{a} + \underline{b} \cdot \underline{b}$$

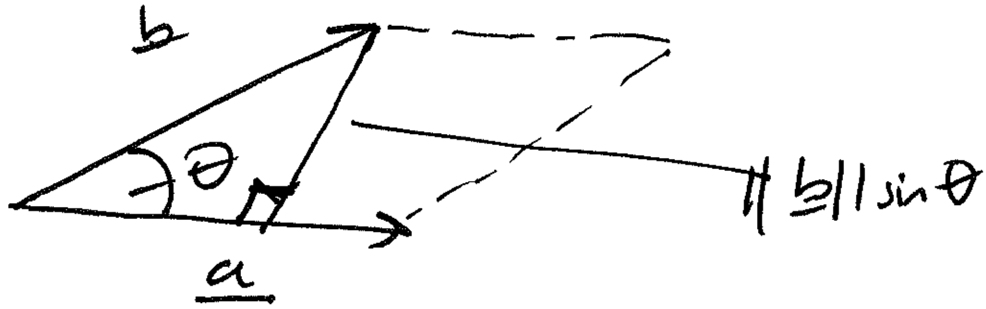
Now, $\underline{a} \cdot b = \underline{b} \cdot \underline{a}$.

and $\boxed{\underline{a} \cdot \underline{b} = 0 \text{ since triangle is has a right angle.}}$

Thus $\|\underline{a}\|^2 + \|\underline{b}\|^2 = \|\underline{c}\|^2$

i.e. $a^2 + b^2 = c^2$. ∎

Let $\underline{a}$ and $\underline{b}$ be non-zero vectors.



The area of the parallelogram they determine is

$$\|\underline{b}\| \sin\theta \cdot \|\underline{a}\|$$

Let $\underline{n}$ be a unit vector ortogonal to both $\underline{a}$ and $\underline{b}$

as shown

Define $\underline{a} \times \underline{b} = \underbrace{\|\underline{a}\| \, \|\underline{b}\| \sin\theta}_{\text{number}} \; \underline{n}$

If either $\underline{a}$ or $\underline{b} = \underline{0}$ then define $\underline{a} \times \underline{b} = \underline{0}$.

We call $\underline{a} \times \underline{b}$ the <u>vector product of $\underline{a}$ and $\underline{b}$</u>

<u>Properties of the vector product</u>

easy to prove

(VP1). $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$.

(VP2). $\lambda(\underline{a} \times \underline{b}) = (\lambda \underline{a}) \times \underline{b} = \underline{a} \times (\lambda \underline{b})$

where $\lambda \in \mathbb{R}$.

Proved in book then

$$\boxed{(\text{VP3}).\quad \underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}}$$

<u>NB</u> The vector product is <u>not</u> associative.