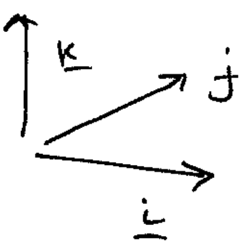
Lecture 26

In te previous lecture, we introduced vectors from a geometric point of view. We now study them from an algebraic point of view. This requires us to introduce co-ordinates.

Choose vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$, all unit vectors, mutually ortogonal where $\underline{k} = \underline{i} \times \underline{j}$.



Each vector $\underline{a}$ can be written uniquely in the form

$$\underline{a} = a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}.$$

We call a_1, a_2, a_3 the <u>components</u> of $\underline{a}$.

We have the following results:

$$\underline{0} = 0\underline{i} + 0\underline{j} + 0\underline{k}.$$

$$\underline{a} + \underline{b} = (a_1 + b_1)\,\underline{i} + (a_2 + b_2)\,\underline{j} + (a_3 + b_3)\underline{k}.$$

$$\lambda \underline{a} = (\lambda a_1)\,\underline{i} + (\lambda a_2)\underline{j} + (\lambda a_3)\underline{k}.$$

Co-ordinate form for the inner product

If $\underline{a} = a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}$ and $\underline{b} = b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k}$.

Then $\boxed{\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3}$

Proof We shall use the following table

$\cdot$	$\underline{i}$	$\underline{j}$	$\underline{k}$
$\underline{i}$	1	0	0
$\underline{j}$	0	1	0
$\underline{k}$	0	0	1

We also use the fact that

$$\underline{x} \cdot (\underline{y} + \underline{z}) = \underline{x} \cdot \underline{y} + \underline{x} \cdot \underline{z}.$$

$$\underline{a} \cdot \underline{b} = (a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}) \cdot \underline{b}$$

$$= (a_1 \underline{i}) \cdot \underline{b} + (a_2 \underline{j}) \cdot \underline{b} + (a_3 \underline{k}) \cdot \underline{b}$$

$$= a_1 (\underline{i} \cdot \underline{b}) + a_2 (\underline{j} \cdot \underline{b}) + a_3 (\underline{k} \cdot \underline{b}).$$

But $\underline{i} \cdot \underline{b} = \underline{i} \cdot (b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k})$

$$= b_1 (\underline{i} \cdot \underline{i}) + b_2 (\underline{i} \cdot \underline{j}) + b_3 (\underline{i} \cdot \underline{k})$$

$$= b_1$$

Similarly, $\underline{j} \cdot \underline{b} = b_2$ and $\underline{k} \cdot \underline{b} = b_3$.

It follows that $\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$, as claimed. ∎

Applications

$$\underline{a} \cdot \underline{a} = a_1^2 + a_2^2 + a_3^2 .$$

Thus $\boxed{\|\underline{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}}$

By definition, $\underline{a} \cdot \underline{b} = \|\underline{a}\| \, \|\underline{b}\| \cos\theta .$

It follows that $\cos\theta = \dfrac{\underline{a} \cdot \underline{b}}{\|\underline{a}\| \; \|\underline{b}\|}$

Thus

$$\boxed{\cos\theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}}}$$

Thus, we can easily compute the angles between vectors in co-ordinate form using the inner product.

Co-ordinate form for the vector product

If $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_2\underline{k}$

then

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \underline{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \underline{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \underline{k}$$

Proof We shall use the following table

$\times$	$\underline{i}$	$\underline{j}$	$\underline{k}$
$\underline{i}$	$\underline{0}$	$\underline{k}$	$-\underline{j}$
$\underline{j}$	$-\underline{k}$	$\underline{0}$	$\underline{i}$
$\underline{k}$	$\underline{j}$	$-\underline{i}$	$\underline{0}$

We shall also use the fact that

$$\underline{x} \times (\underline{y} + \underline{z}) = \underline{x} \times \underline{y} + \underline{x} \times \underline{z},$$

$$\underline{a} \times \underline{b} = (a_1\underline{i} + a_2\underline{j} + a_3\underline{k}) \times \underline{b}$$

$$= a_1(\underline{i} \times \underline{b}) + a_2(\underline{j} \times \underline{b}) + a_3(\underline{k} \times \underline{b}).$$

But

- $\underline{i} \times \underline{b} = \underline{i} \times (b_1\underline{i} + b_2\underline{j} + b_3\underline{k})$

$$= b_1(\underline{i} \times \underline{i}) + b_2(\underline{i} \times \underline{j}) + b_3(\underline{i} \times \underline{k})$$

$$= b_2\underline{k} + b_3(-\underline{j})$$

$$= b_2\underline{k} - b_3\underline{j}$$

- $\underline{j} \times \underline{b} = b_1(\underline{j} \times \underline{i}) + b_2(\underline{j} \times \underline{j}) + b_3(\underline{j} \times \underline{k})$

$$= -b_1\underline{k} + b_3\underline{i}$$

- $\underline{k} \times \underline{b} = b_1(\underline{k} \times \underline{i}) + b_2(\underline{k} \times \underline{j}) + b_3(\underline{k} \times \underline{k}) = b_1\underline{j} - b_2\underline{i}$

Thus

$$\underline{a} \times \underline{b} = a_1(b_2\underline{k} - b_3\underline{j}) + a_2(-b_1\underline{k} + b_3\underline{i}) + a_3(b_1\underline{j} - b_2\underline{i})$$

$$= a_1 b_2 \underline{k} - a_1 b_3 \underline{j} - a_2 b_1 \underline{k} + a_2 b_3 \underline{i} + a_3 b_1 \underline{j} - a_3 b_2 \underline{i}$$

$$= (a_2 b_3 - a_3 b_2)\underline{i} - (a_1 b_3 - a_3 b_1)\underline{j} + (a_1 b_2 - a_2 b_1)\underline{k}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \underline{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \underline{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \underline{k}$$

$\blacksquare$

The scalar triple product

new

$$[\underline{a}, \underline{b}, \underline{c}] = \underline{a} \cdot (\underline{b} \times \underline{c}).$$

Using our results above, we deduce that

$$[\underline{a}, \underline{b}, \underline{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

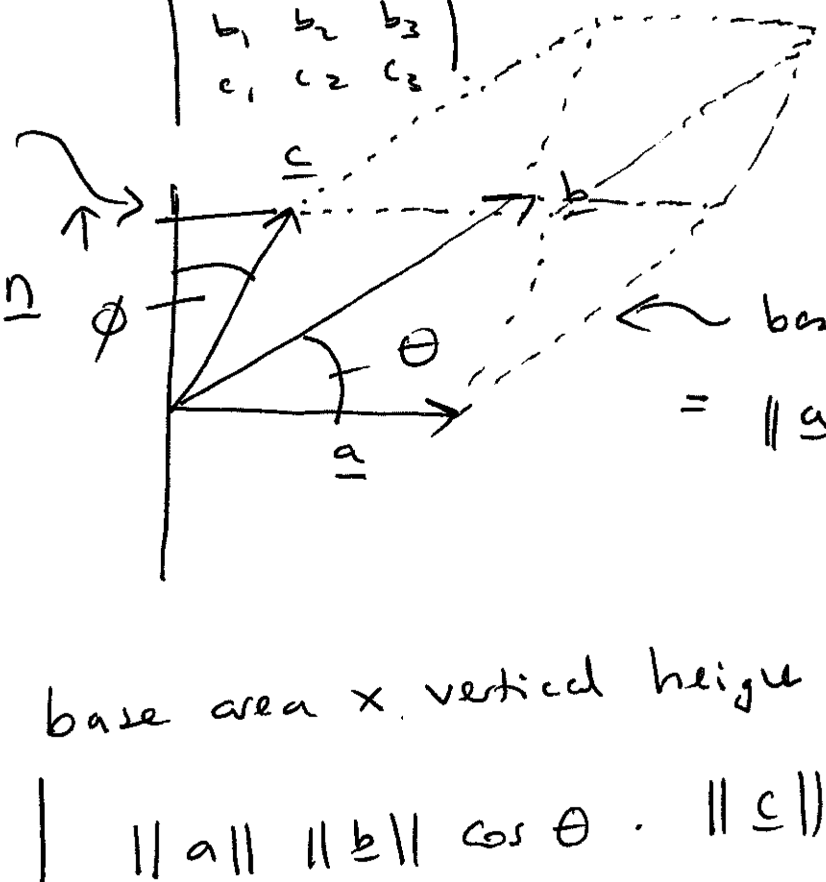
Geometric meaning of determinants

Theorem Let $\underline{a} = a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}$, $\underline{b} = b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k}$ and $\underline{c} = c_1 \underline{i} + c_2 \underline{j} + c_3 \underline{k}$. Then the volume of the parallelepiped (= "squashed box") determined by $\underline{a}, \underline{b}, \underline{c}$ is the absolute value of the determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vertical height = $\big|\, \|\underline{c}\| \cos\phi \,\big|$

Proof



base area $= \|\underline{a}\|\,\|\underline{b}\| \sin\theta$

$$\begin{aligned} \text{Volume} &= \text{base area} \times \text{vertical height} \\ &= \Big|\, \|a\|\,\|\underline{b}\| \cos\theta \cdot \|\underline{c}\| \sin\phi \,\Big| \\ &= \big|\, \underline{c} \cdot (\underline{a} \times \underline{b}) \big| = \big|\, [\underline{c}, \underline{a}, \underline{b}] \big| \end{aligned}$$

We now use the result that if you interchange two rows of a matrix its determinant is multiplied by -1. Thus

$$[\underline{c}, \underline{a}, \underline{b}] = -[\underline{a}, \underline{c}, \underline{b}] = [\underline{a}, \underline{b}, \underline{c}]. \quad \blacksquare$$

Examples

Let $\underline{a} = \underline{i} + 2\underline{j} + 3\underline{k}$,

$\underline{b} = \underline{i} - \underline{j} + 2\underline{k}$,

$\underline{c} = 2\underline{i} + 3\underline{j} + 4\underline{k}$.

1. Calculate $\|\underline{a}\|$.
2. Calculate $\|\underline{b}\|$.
3. Calculate $\underline{a} \circ \underline{b}$.
4. Calculate the angle between $\underline{a}$ and $\underline{b}$ to the nearest degree.
5. Calculate $\underline{a} \times \underline{b}$.
6. Calculate $[\underline{a}, \underline{b}, \underline{c}]$.
7. Calculate $\begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 1 & -1 & 2 \end{vmatrix}$

Solutions

1. $\sqrt{14}$. 2. $\sqrt{6}$. 3. 5. 4. 57°.

5. $7\underline{i} + \underline{j} - 3\underline{k}$ 6. 5. 7. 5.