Lecture 27

Position vectors (= bound vectors)

The vectors we have studied so far are often called free vectors.

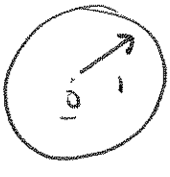
Now choose a fixed point O, called the origin.

A position vector is a vector rooted at O. Position vectors enable us to describe specific points — that is, give their coordinates in vector form. The general position vector is denoted by

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}.$$

Curves can be described in two ways — parametric and non-parametric (parametric means uses a parameter)

Example



← unit circle in the plane.

non-parametric

$x^2+y^2=1$

parametric

$(\cos\theta, \sin\theta)$

where $0 \le \theta < 2\pi$

where θ is the parameter.

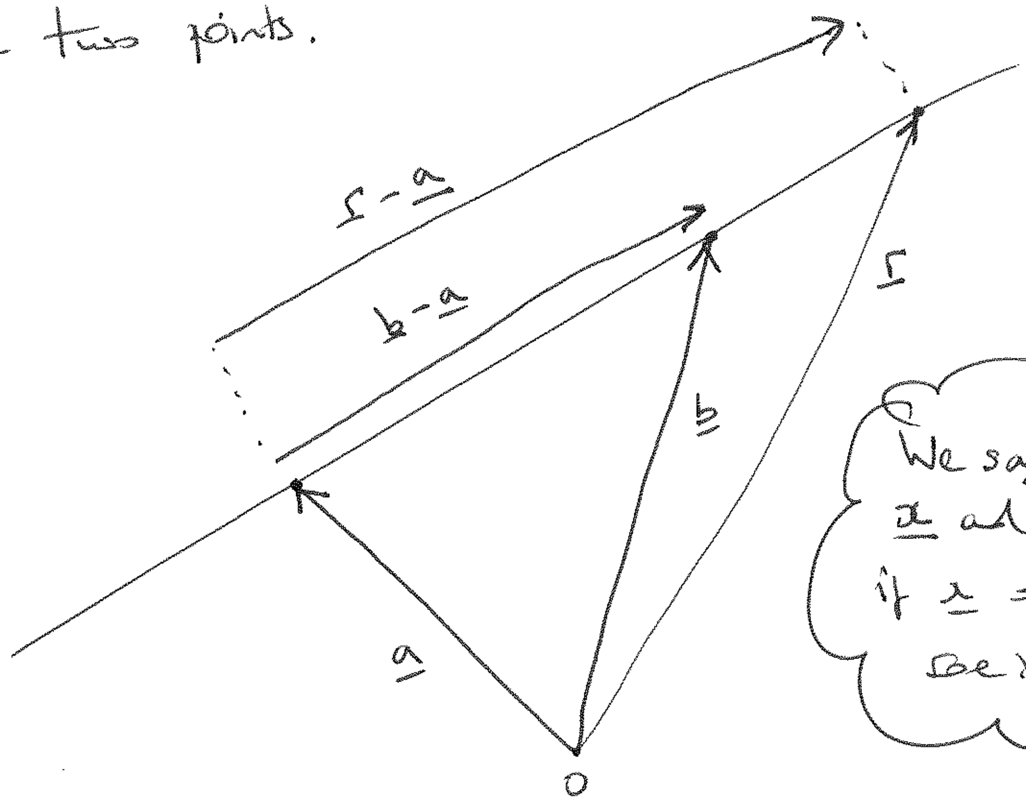
Cartesian coordinates = those given by x, y, z

(Descartes)

Equation of the line

Given: the position vectors $\underline{a}$ and $\underline{b}$ of two distinct points.

Find: the equation of the unique line that contains these two points.



We say vectors $\underline{x}$ and $\underline{y}$ are **parallel** if $\underline{x} = \lambda \underline{y}$ for some $\lambda \in \mathbb{R}$

$\underline{r}$ = position vector of arbitrary point on the line.

$$\underline{r} - \underline{a} = \lambda(\underline{b} - \underline{a}) \quad \text{for some } \lambda \in \mathbb{R}$$

$$\therefore \quad \boxed{\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a})}$$

This is the vector equation of the line. Observe that $\underline{b} - \underline{a}$ is a free vector parallel to the line. Call it $\underline{c}$

$$\underline{r} = \underline{a} + \lambda \underline{c}$$

$$x\underline{i} + y\underline{j} + z\underline{k} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k} + \lambda(c_1\underline{i} + c_2\underline{j} + c_3\underline{k})$$

Equate components

$$\left.\begin{aligned} x &= a_1 + \lambda c_1 \\ y &= a_2 + \lambda c_2 \\ z &= a_3 + \lambda c_3 \end{aligned}\right\} \text{parametric Cartesian equations}$$

$$\therefore \quad \frac{x - a_1}{c_1} = \lambda, \quad \frac{y - a_2}{c_2} = \lambda, \quad \frac{z - a_3}{c_3} = \lambda$$

(Assuming $c_1, c_2, c_3 \neq 0$).

Thus $\dfrac{x - a_1}{c_1} = \dfrac{y - a_2}{c_2}$ and $\dfrac{y - a_2}{c_2} = \dfrac{z - a_3}{c_3}$

$$\therefore \quad c_2(x - a_1) = c_1(y - a_2) \quad \text{and} \quad c_3(y - a_2) = c_2(z - a_3)$$

$$\boxed{c_2 x - c_1 y = c_2 a_1 - c_1 a_2 \quad \text{and} \quad c_3 y - c_2 z = c_3 a_2 - c_2 a_3}$$

Non-parametric equations of the plane: two linear equations

Example Find the vector parametric equation, the Cartesian parametric equation and the non-parametric Cartesian equation of the line determined by the points with position vectors

$\underline{a} = \underline{i} + 2\underline{j} + 3\underline{k}$ and $\underline{b} = 4\underline{i} + \underline{k}$

The vector parametric equation

$$\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a}) \quad \text{so,}$$

$$\boxed{\underline{r} = \underline{i} + 2\underline{j} + 3\underline{k} + \lambda(3\underline{i} - 2\underline{j} - 2\underline{k})}$$

The parametric vector equation

The Cartesian parametric equation

$$x\underline{i} + y\underline{j} + z\underline{k} = (1 + 3\lambda)\underline{i} + (2 - 2\lambda)\underline{j} + (3 - 2\lambda)\underline{k}$$

Now equate components

$$\boxed{x = 1 + 3\lambda, \quad y = 2 - 2\lambda, \quad z = 3 - 2\lambda}$$

parametric Cartesian equation

Check

Both points $\underline{a}$ and $\underline{b}$ lie on both these two lines

$$\frac{x-1}{3} = \lambda, \quad \frac{y-2}{-2} = \lambda, \quad \frac{z-3}{-2} = \lambda$$

Thus $\dfrac{x-1}{3} = \dfrac{2-y}{2}, \quad \dfrac{2-y}{2} = \dfrac{3-z}{2}$

Tidy up to get

$$\boxed{2x + 3y = 8, \quad y - z = -1}$$

Non-parametric Cartesian equation of the line.