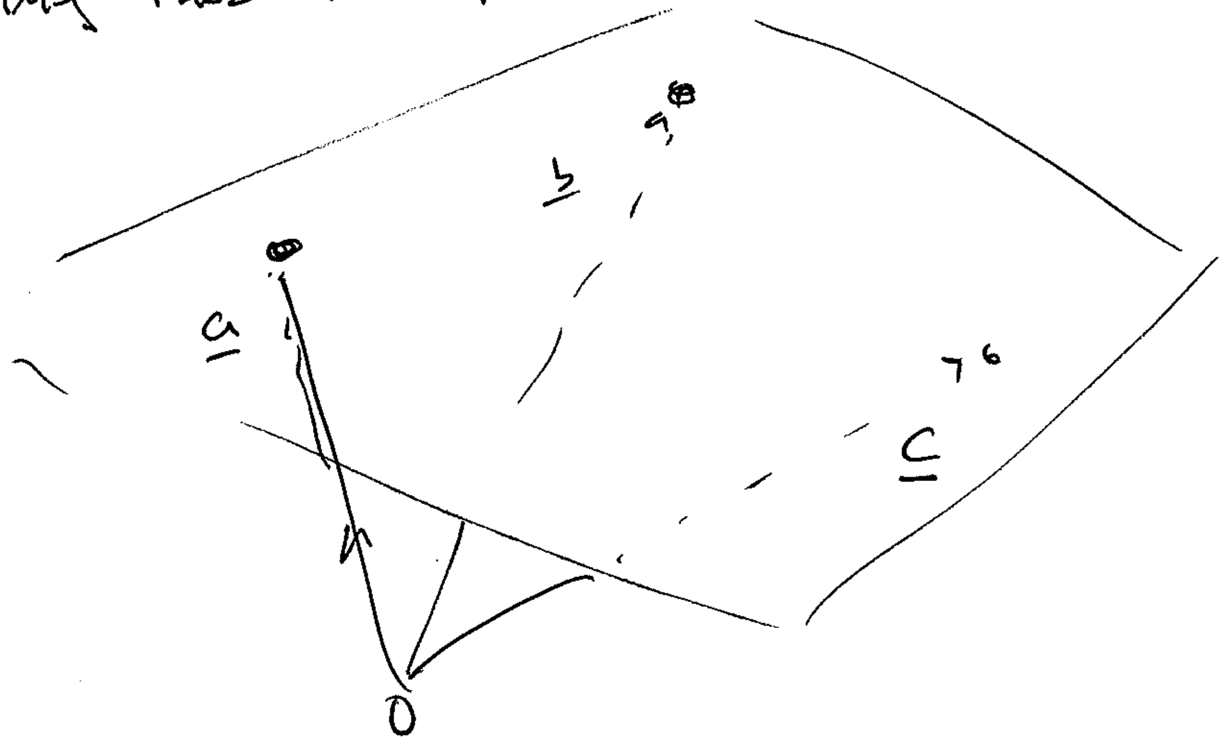
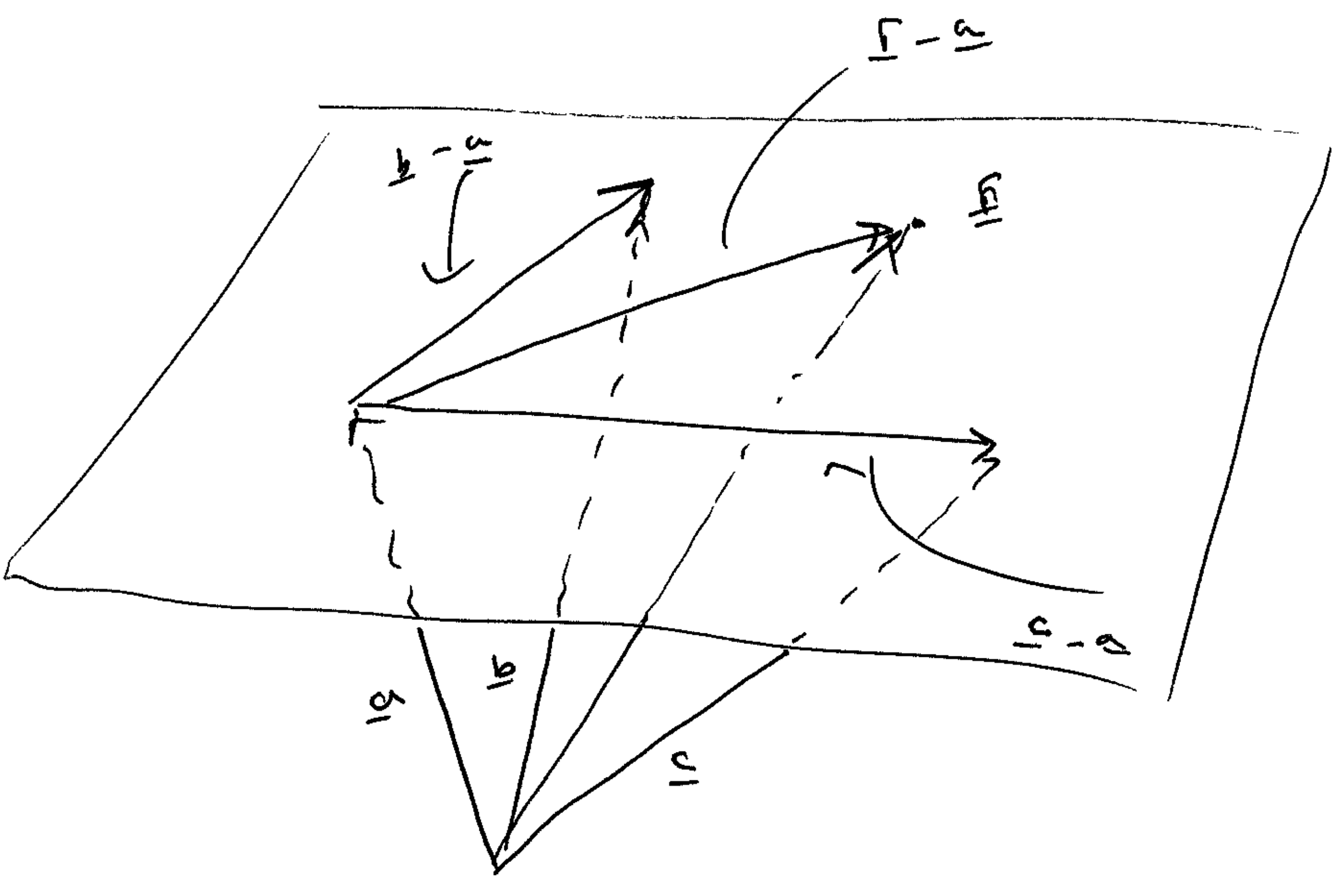
Lecture 28

Equation of the plane

<u>Given:</u> The position vectors of three points $\underline{a}, \underline{b}, \underline{c}$.

<u>Find:</u> The equation of the unique plane containing those three points





$$\underline{r} - \underline{a} = \lambda(\underline{b} - \underline{a}) + \mu(\underline{c} - \underline{a}) \quad \text{for some } \lambda, \mu \in \mathbb{R}$$

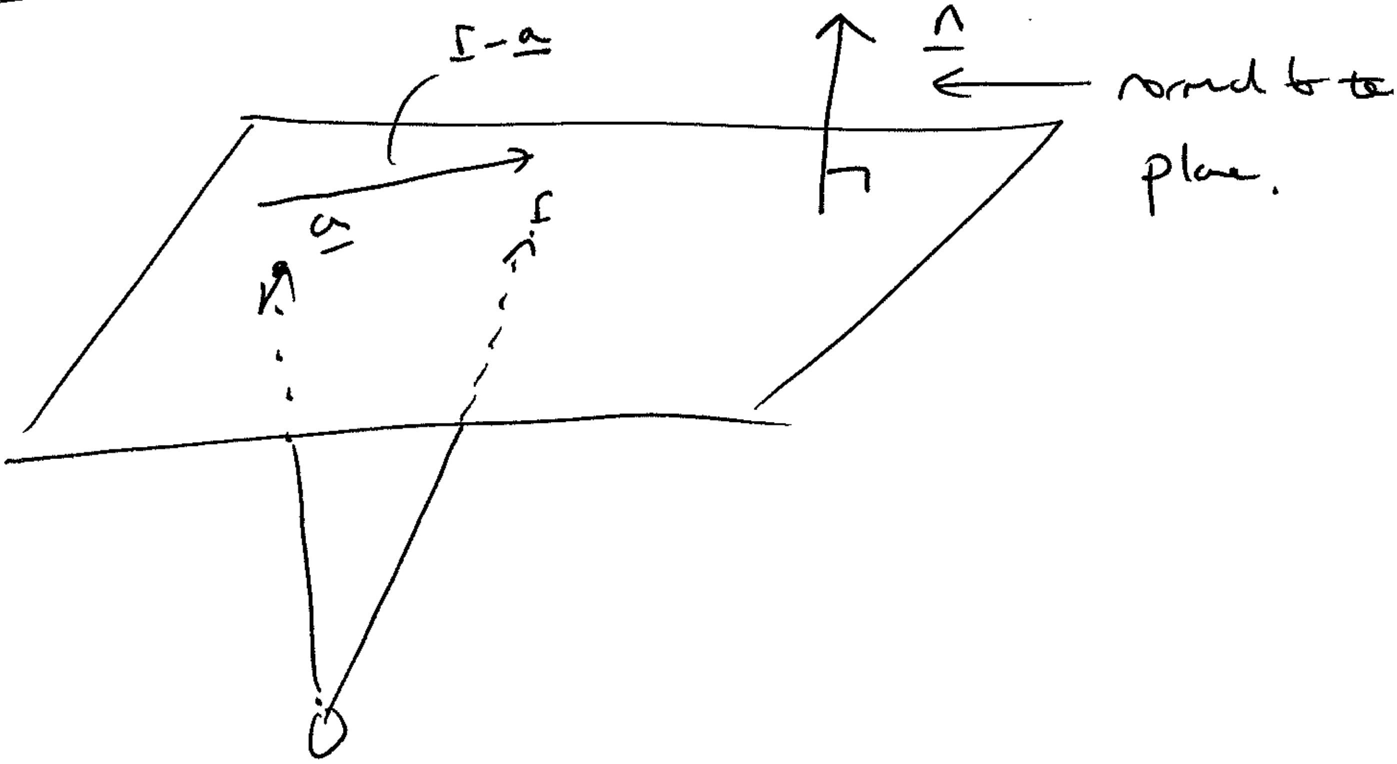
parameters

$$\therefore \quad \boxed{\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a}) + \mu(\underline{c} - \underline{a})}$$

parametric vector equation of the line det'd by $\underline{a}, \underline{b}, \underline{c}$

Observe that $\underline{b}-\underline{a}$ and $\underline{c}-\underline{a}$ are vectors parallel to the plane determined by $\underline{a}, \underline{b}, \underline{c}$.

<u>To obtain the non-parametric Cartesian</u>
<u>equation of the line.</u> We take a different tack.



<u>Given</u>: A position vector of point in the plane and a vector <u>normal</u> to the plane (i.e. orthogonal to it)

Observe that

$$(\underline{r}-\underline{a})\cdot\underline{n}=0.$$

i.e. $$\underline{r}\cdot\underline{n}=\underline{a}\cdot\underline{n}.$$

We can now write down the non-parametric Cartesian equation of the plane:

$$\boxed{n_1x+n_2y+n_3z=a_1n_1+a_2n_2+a_3n_3}$$

This is a linear equation of the form

$$\boxed{ax+by+cz=d}$$

How to find the normal to a plane

Suppose the plane is defined by the three position vectors $\underline{a}, \underline{b}, \underline{c}$. Then

$$\underline{n} = (\underline{b} - \underline{a}) \times (\underline{c} - \underline{a})$$

is a normal.

Systems of linear equations in 3 unknowns revisited

The equation $ax + by + cz = d$

represents a plane, in general, except

- If $a = b = c = d = 0$ then it is all $\mathbb{R}^3$.
- If $a = b = c = 0$, $d \neq 0$ where it is $\emptyset$

<u>In general</u> the intersection of two planes will be a line. This is how we described lines in a non-parametric way.

<u>In general</u> the intersection of three planes will be a point.

But in both the above cases there are exceptions depending on how the planes are angled.

Example Determine the parametric and non-parametric equations of the plane determined by the following three points:

$$\underline{a} = \underline{i} - \underline{j} + 3\underline{k}$$
$$\underline{b} = 4\underline{i} + \underline{j} - 2\underline{k}$$
$$\underline{c} = -\underline{i} - \underline{j} + \underline{k}$$

$$\underline{b} - \underline{a} = 3\underline{i} + 2\underline{j} - 5\underline{k}$$
$$\underline{c} - \underline{a} = -2\underline{i} + 0\underline{j} - 2\underline{k}$$

The parametric equation is given by

$$\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a}) + \mu(\underline{c} - \underline{a}).$$

Thus

$$\underline{r} = (\underline{i} - \underline{j} + 3\underline{k}) + \lambda(3\underline{i} + 2\underline{j} - 5\underline{k}) + \mu(-2\underline{i} + 0\underline{j} - 2\underline{k})$$

The non-parametric Cartesian equation

We need to compute the normal

$$\underline{n} = (\underline{b} - \underline{a}) \times (\underline{c} - \underline{a}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 2 & -5 \\ -2 & 0 & -2 \end{vmatrix}$$

$$= -4\underline{i} + 16\underline{j} + 4\underline{k}$$

Check

$\underline{n} \cdot (\underline{b} - \underline{a}) =$ ~~$-4 + 16$~~ $-12 + 32 - 20 = 0$ ✓

$\underline{n} \cdot (\underline{c} - \underline{a}) = 8 + 0 - 8 = 0$ ✓

The required equation is $(\underline{r} - \underline{a}) \cdot \underline{n} = 0$

or $\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}$. $\left(= (\underline{0} - \underline{j} + 3\underline{k}) \cdot (-4\underline{i} + 16\underline{j} + 4\underline{k}) \right.$

$= -4 - 16 + 12$

This gives $\boxed{-4x + 16y + 4z = -8}$

or $\boxed{x - 4y - z = 2}$

Check the three original points satisfy this equation.