

## Lecture 29

### Proof by induction

Useful proof technique, but given undue prominence in school maths because it is ~~a~~ fairly routine.

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### Well ordering principle

Every non-empty subset of  $\mathbb{N}$  has a smallest element.

We use this ~~&~~ principle to prove the following:

Induction principle Let  $X \subseteq \mathbb{N}$  be s.t.

$$(1) 0 \in X.$$

$$(2) n \in X \Rightarrow n+1 \in X.$$

$$\text{Then } X = \mathbb{N}.$$

Proof Suppose that  $\mathbb{N} \neq X$ .

Then  $\mathbb{N} \setminus X \neq \emptyset$ .

By the well ordering principle, this set has a smallest element, we call it  $m$ .

Observe  $m \neq 0$  since by (1)  $0 \in X$ .

It follows that  $m-1 \in X$ .

But by (2),  $m \in X$ . Contradiction.

It follows that  $\mathbb{N} \setminus X = \emptyset$  and so  $\mathbb{N} = X$ .



Application We want to prove

that infinitely many statements

$S_0, S_1, S_2, \dots$

are all true.

Let  $X \subseteq \mathbb{N}$  be defined as follows:

$i \in X \iff S_i$  is true.

If  $X$  satisfies the induction principle +

$X = \mathbb{N}$  then  $S_i$  is true for all  $i \geq 0$ .

The proof that  $S_0, S_1, S_2, \dots$

are all true is organized as follows:

(1) Base case. Prove that  $S_0$  is true

(2) IH (Induction hypothesis). Assume that  $S_n$  is true.

On the empty set, prove that  $S_{n+1}$  is true.

By means of ~~the~~ ~~an~~ similar arguments,  
the induction principle case extended.

Define  $\mathbb{N}^{\geq n} = \{n, n+1, n+2, \dots\}$

Then if  $X \subseteq \mathbb{N}^{\geq n}$  is sk.

$$(1) n \in X$$

$$(2) n \in X \Rightarrow n+1 \in X$$

$$\text{Then } X = \mathbb{N}^{\geq n}.$$


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# Examples

When  $n \geq 1$  we have De Moivre's theorem

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$


Proof When  $n=1$  result is clearly true

Assume result holds for  $n$ .

We prove that it now holds for  $n+1$ .

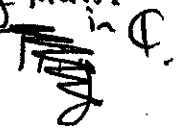
$$(\cos \theta + i \sin \theta)^{n+1} =$$

$$(\cos \theta + i \sin \theta) (\cos \theta + i \sin \theta)^n$$

IH 

$$= (\cos \theta + i \sin \theta) (\cos n\theta + i \sin n\theta)$$

$$= \cos (\theta + n\theta) + i \sin (\theta + n\theta)$$

properties of mult. in  $\mathbb{C}$  

$$= \cos (n+1)\theta + i \sin (n+1)\theta \quad \square$$

A non-constant polynomial of degree  $n$  has at most  $n$  roots.

Proof. Check the case  $n = 1$

same result is true for all poly of degree  $n$ .

Let  $f(x)$  be a poly of degree  $n+1$ .

If  $f(x)$  has no roots we are done.

If  $f(x)$  has a root  $a$  then

$$f(x) = (x-a)g(x)$$

where  $\deg g(x) = n$ .

By (IH)  $g(x)$  has at most  $n$  roots.

Thus  $f(x)$  has at most  $n+1$  roots.  $\square$

$$\overline{z_1 + \dots + z_n} = \overline{z_1} + \dots + \overline{z_n}$$

Proof when  $n=2$

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

Let  $z_1 = a_1 + b_1 i$        $z_2 = a_2 + b_2 i$

$$\therefore z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2) i$$

$$\therefore \overline{z_1 + z_2} = (a_1 + a_2) - (b_1 + b_2) i$$

$$\overline{z_1} + \overline{z_2} = (a_1 - b_1 i) + (a_2 - b_2 i)$$

$$= (a_1 + a_2) - (b_1 + b_2) i$$

$$\therefore \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

using the result  
for  $n=2$

Induction. Assume result true for  $n$ . Prove for  $n+1$ .

$$\begin{aligned} \overline{(z_1 + \dots + z_n + z_{n+1})} &= \overline{(z_1 + \dots + z_n) + z_{n+1}} \\ &= \overline{(z_1 + \dots + z_n)} + \overline{z_{n+1}} \end{aligned}$$

IH used here.

$$= (\bar{z}_1 + \dots + \bar{z}_n) + \bar{z}_{n+1} \quad \square$$

$$\overline{(z_1, \dots, z_n)} = \bar{z}_1, \dots, \bar{z}_n$$

is proved in a similar way.

You can find out more  
 about proof by induction  
 in Section 3.8 of my book.