Lecture 3

Let A be a set.

Define

$$P(A) = \{B : B \subseteq A\}$$

is called the <u>power set of A</u>. It is the set of all subsets of A.

<u>Example</u> Let $A = \{1, 2, 3\}$. We calculate $P(A)$.

Cardinality of subset	Subsets
0	$\emptyset$
1	$\{1\}, \{2\}, \{3\}$
2	$\{1,2\}, \{1,3\}, \{2,3\}$
3	$\{1,2,3\}$

It follows that

$$P(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

Observe that

$$|P(\{1,2,3\})| = 8 = 2^3.$$

There is an interesting pattern in the number of subsets of a given size

Cardinality of subsets	number of such subsets
0	1
1	3
2	3
3	1

So,

$$8 = 1 + 3 + 3 + 1$$

These numbers occur in Pascal's triangle

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \leftarrow ! \end{array}$$

This is not a coincidence.

$$\boxed{\text{How does the power set "know" about Pascal's triangle?}}$$

Some questions

(1) What is the relationship between $|P(X)|$ and $|X|$?

(2) If $A \subseteq X$ and $|A| = k$.

also called a combination

We call A a **k-subset**.

How many k-subsets are there of an n-element set?

Theorem 1

$$|P(X)| = 2^{|X|}$$

Examples

(1) $X = \{1\}$

$P(X) = \{\emptyset, \{1\}\}$.

$\therefore \quad |P(\{1\})| = 2 = 2^{1}$

(2) $X = \{1, 2\}$

$P(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

$\therefore \quad |P(\{1,2\})| = 4 = 2^{2}$

Define $0! = 1$

$n! = n(n-1)!$ when $n \geq 1$

<u>Example</u> $3! = 3.2.1 = \underline{\underline{6}}$

Calculator

Define ${}^{n}C_{k} = \binom{n}{k} = \dfrac{n!}{k!(n-k)!}$

when $0 \leq k \leq n$

n Choose k

called a <u>binomial coefficient</u>

<u>Theorem 2</u> Let $0 \leq k \leq n$.

Then the number of k-subsets of an n-element set is $\binom{n}{k}$.

K-subset is sometimes called a combination.

Example Let $n = 3$.

$$\binom{3}{0} = \frac{3!}{0!(3-0)!} = 1$$

$$\binom{3}{1} = \frac{3!}{1!\ 2!} = \frac{3 \cdot 2!}{2!} = 3$$

$$\binom{3}{2} = \frac{3!}{2!\,1!} = 3$$

$$\binom{3}{3} = \frac{3!}{3!\,0!} = 1$$

Example Let $X = \{a, b, c, d\}$. We find all 2-subsets: $\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}$ 6 such subsets. $\binom{4}{2} = \frac{4!}{2!\,2!}$

$$= \frac{4 \cdot 3}{2} = 2 \cdot 3 = 6$$

<u>Example</u> In the National lottery, you have to choose 6 numbers from 59. Thus the number of ways of doing this is $\binom{59}{6}$

$$= 45,057,474$$

<u>Example</u> A committee is a set of people. Then number of 4-people committees from 140 people (students, the people in this lecture) is

$$\binom{140}{4}$$

Define

$$\sum_{i=0}^{n} x_i = x_0 + x_1 + \dots + x_n$$

Theorem 3 Let $n \geqslant 0$.

$$2^n = \sum_{i=0}^{n} \binom{n}{k}$$

This theorem can be proved by counting.

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Notation

$\{\,\}$, $\in$, $\notin$, $\emptyset$, $|X|$

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$

$\subseteq$, $P(X)$, $\{a : \text{property of } a\}$

Words

Set, element, equality of sets, is an element of, cardinality, natural numbers, integers, rational numbers, real numbers, subset, power set, k-subset.