Lecture 7

Revision of Section 1

1. (i) How many ~~elements~~ subsets! does a 15 element set have?

   (ii) How many of those subsets have exactly 7 elements?

   (iii) How many of those subsets have exactly 8 elements?

   (iv) What is the connection between the numbers you obtained in (ii) & (iii)? Explain.

2. How many registration plates can be made using two (upper case) letters followed by a 3-digit number (not beginning with 0)?

3. (i) How many 4-digit numbers can be made from the set of digits $\{1, 2, 3, 4, 5\}$ if repetitions are allowed?

   (ii) Repeat (i) but this time no repetitions.

   (iii) Repeat (ii) but add the requirement that the numbers be odd. A number is odd if it ends in a digit which is odd

   not the digits

4. A coin and a dice are thrown. Describe the set of possible outcomes.

5. Five coins are thrown. How many outcomes contain exactly 2 heads?

6. Draw Venn diagrams to illustrate

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

7. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

8. Prove that $|A \setminus B| = |A| - |A \cap B|$.

9. Prove that $|A \cup B| = |A| + |B| - |A \cap B|$.

10. Define the binary operation $\circ$ on $\mathbb{R}$ by

$$a \circ b = a + b + ab.$$

(i) Is $\circ$ commutative?

(ii) Is $\circ$ associative?

1. (i) 2^{15} (ii) $\binom{15}{7}$ (iii) $\binom{15}{8}$

(iv) $\binom{15}{7} = \binom{15}{8}$ since to choose 7 objects from 15 is to reject 8 objects from 15.

2. Let $\mathcal{A} = \{A, B, \ldots, Z\}$ and $\mathcal{D} = \{0, 1, \ldots, 9\}$.
The registration plates are therefore elements of the set

$$\mathcal{A}^2 \times \mathcal{D}\setminus\{0\} \times \mathcal{D} \times \mathcal{D}$$

(a 5-tuple). The number of such registration plates is therefore $26^2 \times 9 \times 10^2 = \underline{\underline{608,400}}$

3. (i) $|\{1,2,3,4,5\}^4| = 5^4 = \underline{\underline{625}}$

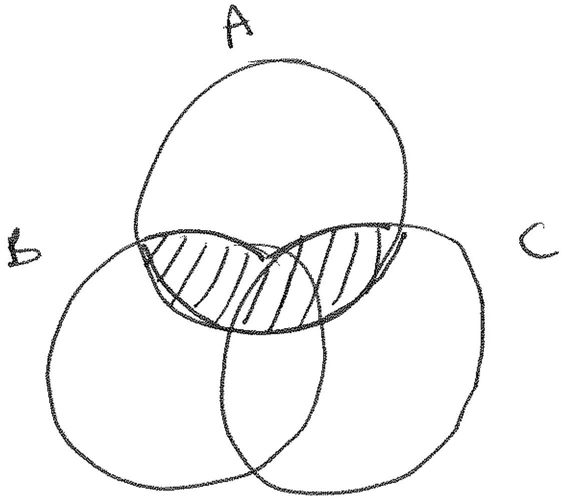
(ii) ${}^5P_4 = \underline{\underline{120}}$.

(iii) ${}^4P_3 \times 3 = \underline{\underline{72}}$

4. $\{H, T\} \times \{1, 2, 3, 4, 5, 6\}$.

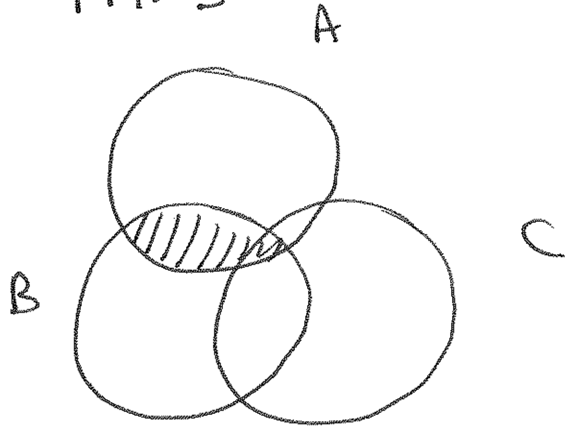
5. $\binom{5}{2}$.

6. $A \cap (B \cup C)$

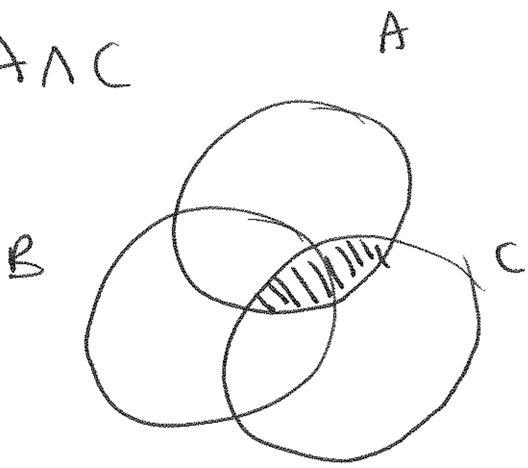


The same Venn diagrams

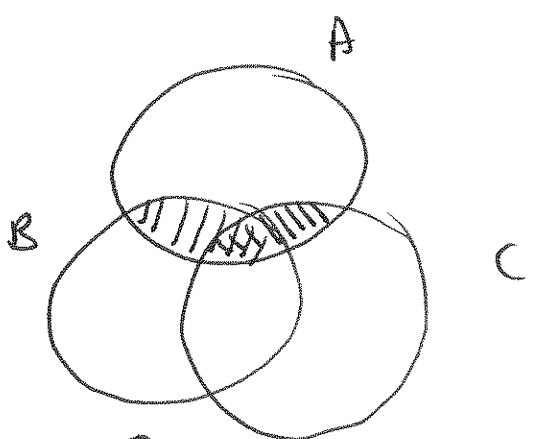
$A \cap B$



$A \cap C$



$(A \cap B) \cup (A \cap C)$



7. We have to prove two results:

(i) $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

(ii) $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

<u>Proof of (i)</u> Let $x \in A \cap (B \cup C)$. Then $(x \in A)$ and $(x \in B \cup C)$.

$x \in B \cup C$ means $(x \in B)$ or $(x \in C)$ or $((x \in B)$ and $(x \in C))$

Thus $x \in A \cap (B \cup C)$ means

We prove using cases

- $(x \in A)$ and $(x \in B)$

or

- $(x \in A)$ and $(x \in C)$

or

- $(x \in A)$ and $(x \in B \cap C)$.

That is $x \in A \cap B$

or

$x \in A \cap C$

or

$x \in A \cap (\cancel{B \cap C})$ $(\subseteq A \cap B, A \cap C)$

Thus it means $\underline{x \in (A \cap B) \cup (A \cap C).}$

<u>Proof of (ii)</u> Let $x \in (A \cap B) \cup (A \cap C)$.

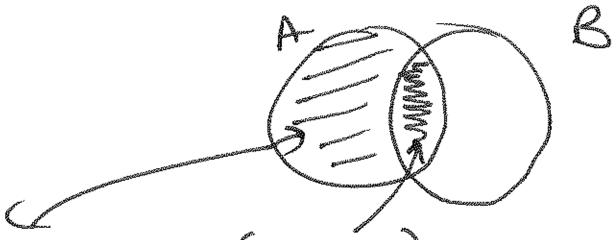
Then $(x \in A \cap B)$ or $(x \in A \cap C)$

or $x \in (A \cap B) \cancel{\cap (A} \cap C) \subseteq A \cap B, A \cap C$.

So, $(x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

So, $x \in A$ and $((x \in B)$ or $(x \in C))$

This gives $\underline{x \in A \cap (B \cup C)}$.



8. $A = A \setminus B \cup (A \cap B)$

(This can be proved as in Q7).

But $A \setminus B$ and $A \cap B$ are <u>disjoint</u>.

$\therefore \quad |A| = |A \setminus B| + |A \cap B|$

So, $\quad |A \setminus B| = |A| - |A \cap B|$.

9. $A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$

This is a disjoint union. So,

$|A \cup B| = |A \setminus B| + |A \cap B| + |B \setminus A|$

By Q8, $\quad |A \setminus B| = |A| - |A \cap B|$

$|B \setminus A| = |B| - |A \cap B|$

So,

$$|A \cup B| = |A| - \cancel{|A \cap B|}$$
$$+ \cancel{|A \cap B|}$$
$$+$$
$$|B| - |A \cap B|$$

$$= |A| + |B| - |A \cap B|.$$

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10. (i) $a \circ b = a + b + ab$

$b \circ a = b + a + ba$

But $a + b = b + a$ and $ab = ba$ so $a \circ b = b \circ a$. It follows that $\circ$ <u>is</u> commutative.

(ii) $a \circ (b \circ c) = a \circ (b + c + bc)$

$= a + (b + c + bc) + a(b + c + bc)$

$= a + b + c + bc + ab + ac + abc$

$(a \circ b) \circ c = (a + b + ab) \circ c$

$= (a + b + ab) + c + (a + b + ab)c$

$= a + b + ab + c + ac + bc + abc$

$= (a + b + c) + (ab + ac + bc) + abc$

~~By~~ Using the commutativity and associativity of $+$ we see that $a \circ (b \circ c) = (a \circ b) \circ c$ and so $\circ$ is associative.