Lecture 9

Axioms for highschool algebra

(Think $\mathbb{R}$).

Addition

(F1) Addition is associative:

$$(x+y)+z = x+(y+z).$$

(F2) There is an additive identity:

$$x+0 = x = 0+x.$$

(F3) Each number has an additive inverse: for each x there is a unique $-x$ s.t.

$$x+(-x) = 0 = (-x)+x$$

(F4) Addition is commutative:

$$x+y = y+x.$$

<u>Definition</u> $\boxed{x-y = x+(-y).}$

Multiplication

(F5) Multiplication is associative:

$$x(yz) = (xy)z.$$

(F6) There is a multiplicative identity

$$1x = x = x1.$$

(F7) Each <u>non-zero</u> number has a multiplicative inverse. That is, for each $x \neq 0$ there exists a unique x^{-1} s.t. $x x^{-1} = 1 = x^{-1} x$

(F8) Multiplication is commutative.

<u>Definition</u> $x \div y = x y^{-1}$

(when $y \neq 0$).

Linking axioms

(F9) $0 \neq 1$.

(F10) The additive identity is a multiplicative zero. That is $0x = 0 = x0$.

(F11) Multiplication distributes over addition.

$$x(y+z) = xy + xz$$

$$(y+z)x = yx + zx$$

Any set satisfying these 11 axioms is called a <u>field</u>.

Thus $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ (later!) are all fields.

<u>Example</u> Solve $ax=b$ when $a\neq 0$.

$$ax=b$$

$$a^{-1}(ax) = a^{-1}b \quad \text{by (F7)}$$

$$(a^{-1}a)x = a^{-1}b \quad \text{by (F5)}$$

$$1x = a^{-1}b \quad \text{by (F7)}$$

$$\underline{x = a^{-1}b} \quad \text{by (F6)}$$

Example $(-1) \times (-1) = 1$

We show that $(-1)(-1) - 1 = 0$.

$$
\begin{aligned}
(-1)(-1) - 1 &= (-1)(-1) + (-1) && \text{def'n} \\
&= (-1)(-1) + (-1)1 && (F6) \\
&= (-1)(-1 + 1) && (F11) \\
&= (-1)0 && \text{by } (F3) \\
&= 0 && \text{by } (F10)
\end{aligned}
$$

Notation Because of the associativity of addition we can write a sum such as

$$x_1 + \dots + x_n$$

without using brackets.

We usually abbreviate $x_1 + \dots + x_m$ to

$$\sum_{i=1}^{m} x_i \qquad \text{or} \qquad \sum_{j=\boxed{1}}^{\boxed{m}} x_j$$

called the limits of the sum. Pay attention to them.

We shall now Convince some of our results from Section 1 with algebra.

Idea (Example) Let's look at all products of x and y which contain 3 letters.

① $x\,x\,x$
② $x\,x\,y$
③ $x\,y\,x$
④ $y\,x\,x$
⑤ $x\,y\,y$
⑥ $y\,x\,y$
⑦ $y\,y\,x$
⑧ $y\,y\,y$

$\therefore$ 8 such 3-tuples $= \left|\{x, y\}^3\right| = \underline{\underline{8}}$

s'pose now we assume associativity and commutativity.

Then $xxy = xyx = yxx$

$\therefore$ we get $\underline{\underline{3}}\, x^2 y$ $\quad \binom{3}{1} \leftarrow$ # ys

$xyy = yxy = yyx$

$\therefore$ we get $\underline{\underline{3}}\, x y^2$ $\quad \binom{3}{2} \leftarrow$ # ys

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This is the idea behind the proof of the following.

Binomial theorem

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

NB

$$= \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

NB

An expression of the form $x+y$ is called a binomial because it is in two parts.
