# QUIZ 2: thought provoking questions

(1) (a) Prove that the last digit in the square of a positive whole number must be one of 0,1,4,5,6 or 9. Is the converse true?
  (b) Prove that a natural number is even if and only if its last digit is even.
  (c) Prove that a natural number is exactly divisible by 9 if and only if the sum of its digits is divisible by 9.

(2) Inscribe a regular polygon with $2^{m+1}$ sides inside the circle with radius 1. That is, the vertices of the polygon lie on the circumference of the circle. Show that

$$2^m \underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\dots}}}}_{m\text{ times}}$$

is an approximation to $\pi$.

(3) In how many ways can a committee consisting of 3 Montagues and 2 Capulets be chosen from 7 Montagues and 5 Capulets?

(4) Define $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. A binary operation $\circ$ defined on $\mathbb{N}^*$ is known to have the following properties.
  (a) $a \circ b = b \circ a$.
  (b) $a \circ a = a$.
  (c) $a \circ (a+b) = a \circ b$.

What is this binary operation?

(5) Let $A$ be an $n \times n$ matrix such that $AB = BA$ for all $n \times n$ matrices $B$. Show that $A = \lambda I$ for some scalar $\lambda$.

(6) In this question, we show that the complex numbers could be defined via (real) matrices. Let $\mathcal{C}$ be the set of all real matrices that have the following shape

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

  (a) Prove that $\mathcal{C}$ is closed under sum, difference and product.
  (b) Prove that matrix multiplication is commutative in $\mathcal{C}$.
  (c) Prove that every non-zero matrix in $\mathcal{C}$ is invertible and that its inverse also belongs to $\mathcal{C}$.
  (d) Deduce that the set $\mathcal{C}$ with these operations is a field.
  (e) It remains to show how $\mathcal{C}$ is related to $\mathbb{C}$. Define

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \mathbf{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Show that we may write our matrices in the form

$$a\mathbf{1} + b\mathbf{i} \text{ where } \mathbf{i}^2 = -\mathbf{1}.$$

(7) Calculate

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}).$$

(8) Prove that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

(9) Let $A$ be an $n \times n$ matrix. The *trace* of $A$, denoted by $\mathrm{tr}(A)$, is defined to be $\sum_{i=1}^{n} (A)_{ii}$.

(a) Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

(b) Let $A$ be a fixed $n \times n$ matrix. Find all solutions to

$$AX - XA = I$$

where $I$ is the $n \times n$ identity matrix.

(c) Let $A$ be a square real matrix. Let $\varepsilon > 0$ be small so that $\varepsilon^2$ is really small. Prove that $\det(I + \epsilon A) \approx 1 + \varepsilon \mathrm{tr}(A)$ where $\approx$ means *approximately equal to*.