The adjugate matrix and the inverse matrix

This is a version of part of Section 8.5.

The adjugate of a square matrix

Let A be a square matrix. By definition, the *adjugate* of A is a matrix B, often denoted by $\operatorname{adj}(A)$, with the property that

$$AB = \det(A)I = BA$$

where I is the identity matrix the same size as A. We shall show how to construct the adjugate of any square matrix.

The 2×2 case

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We construct the adjugate as follows.

- Replace each entry a_{ij} of A by the element remaining when the ith row and jth column are crossed out

$$\begin{pmatrix} d & c \\ b & a \end{pmatrix}.$$

- Use the following matrix of signs

$$\begin{pmatrix} + & - \\ - & + \end{pmatrix},$$

  where the entry in row i and column j is the sign of $(-1)^{i+j}$, to get

$$\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

- Take the transpose of this matrix to get the adjugate of A

$$\operatorname{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

- Observe that

$$A \operatorname{adj}(A) = \det(A)I = \operatorname{adj}(A)\, A.$$

Example Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}.$$

Then

$$\operatorname{adj}(A) = \begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix}.$$

The general case

Let A be an $n \times n$ matrix with entries a_{ij}. We define its adjugate as the result of the following sequence of operations.

- Choose an entry a_{ij} in the matrix A.
- Crossing out the entries in row i and column j, an $(n-1) \times (n-1)$ matrix is constructed, denoted by $M(A)_{ij}$, and called a *submatrix*.
- The determinant $\det(M(A)_{ij})$ is called the *minor* of the element a_{ij}.
- If $\det(M(A)_{ij})$ is multiplied by the corresponding sign, we get the *cofactor* $c_{ij} = (-1)^{i+j} \det(M(A)_{ij})$ of the element a_{ij}.
- Replace each element a_{ij} by its cofactor to obtain the matrix $C(A)$ of cofactors of A.
- The *transpose* of the matrix of cofactors $C(A)$ is the adjugate.

Example Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ -1 & 1 & 2 \end{pmatrix}.$$

The *matrix of minors* is

$$\begin{pmatrix} -1 & 5 & 2 \\ 1 & 5 & 3 \\ 2 & -5 & -4 \end{pmatrix}.$$

The *matrix of cofactors* is

$$\begin{pmatrix} -1 & -5 & 2 \\ -1 & 5 & -3 \\ 2 & 5 & -4 \end{pmatrix}.$$

The *adjugate* is the transpose of the matrix of cofactors

$$\begin{pmatrix} -1 & -1 & 2 \\ -5 & 5 & 5 \\ 2 & -3 & -4 \end{pmatrix}.$$

Theorem *A square matrix A is invertible if and only if* $\det(A) \neq 0$*. If* $\det(A) \neq 0$ *then the inverse of A is given by*

$$A^{-1} = \frac{1}{\det(A)} adj(A).$$

Example The matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

has $\det(A) = -5$. The inverse of A therefore exists and is equal to

$$A^{-1} = -\frac{1}{5} \begin{pmatrix} -1 & -1 & 2 \\ -5 & 5 & 5 \\ 2 & -3 & -4 \end{pmatrix}.$$