The adjugate matrix and the inverse matrix

This is a version of part of Section 8.5.

The adjugate of a square matrix

Let $A$ be a square matrix. By definition, the adjugate of $A$ is a matrix $B$, often denoted by $\text{adj}(A)$, with the property that

$$AB = \det(A)I = BA$$

where $I$ is the identity matrix the same size as $A$. We shall show how to construct the adjugate of any square matrix.

The $2 \times 2$ case

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$ 

We construct the adjugate as follows.

- Replace each entry $a_{ij}$ of $A$ by the element remaining when the $i$th row and $j$th column are crossed out

  $$\begin{pmatrix} d & c \\ b & a \end{pmatrix}.$$ 

- Use the following matrix of signs

  $$\begin{pmatrix} + & - \\ - & + \end{pmatrix},$$

  where the entry in row $i$ and column $j$ is the sign of $(-1)^{i+j}$, to get

  $$\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$ 

- Take the transpose of this matrix to get the adjugate of $A$

  $$\text{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$ 

- Observe that

  $$A \text{adj}(A) = \det(A)I = \text{adj}(A)A.$$ 

Example Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}.$$ 

Then

$$\text{adj}(A) = \begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix}.$$
The general case

Let $A$ be an $n \times n$ matrix with entries $a_{ij}$. We define its adjugate as the result of the following sequence of operations.

- Choose an entry $a_{ij}$ in the matrix $A$.
- Crossing out the entries in row $i$ and column $j$, an $(n-1) \times (n-1)$ matrix is constructed, denoted by $M(A)_{ij}$, and called a submatrix.
- The determinant $\det(M(A)_{ij})$ is called the minor of the element $a_{ij}$.
- If $\det(M(A)_{ij})$ is multiplied by the corresponding sign, we get the cofactor $c_{ij} = (-1)^{i+j} \det(M(A)_{ij})$ of the element $a_{ij}$.
- Replace each element $a_{ij}$ by its cofactor to obtain the matrix $C(A)$ of cofactors of $A$.
- The transpose of the matrix of cofactors $C(A)$ is the adjugate.

Example Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ -1 & 1 & 2 \end{pmatrix}.$$ 

The matrix of minors is

$$\begin{pmatrix} -1 & 5 & 2 \\ 1 & 5 & 3 \\ 2 & -5 & -4 \end{pmatrix}.$$ 

The matrix of cofactors is

$$\begin{pmatrix} -1 & -5 & 2 \\ -1 & 5 & -3 \\ 2 & 5 & -4 \end{pmatrix}.$$ 

The adjugate is the transpose of the matrix of cofactors

$$\begin{pmatrix} -1 & -1 & 2 \\ -5 & 5 & 5 \\ 2 & -3 & -4 \end{pmatrix}.$$ 

Theorem A square matrix $A$ is invertible if and only if $\det(A) \neq 0$. If $\det(A) \neq 0$ then the inverse of $A$ is given by

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A).$$

Example The matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

has $\det(A) = -5$. The inverse of $A$ therefore exists and is equal to

$$A^{-1} = -\frac{1}{5} \begin{pmatrix} -1 & -1 & 2 \\ -5 & 5 & 5 \\ 2 & -3 & -4 \end{pmatrix}.$$