SCHOOL OF MATHEMATICAL AND COMPUTER SCIENCES

Department of Mathematics

F17LP

Logic and Proof

Semester 1 – 2019/20

Duration: 2 Hours

Attempt all questions

A University approved calculator may be used
for basic computations, but
appropriate working must be shown to obtain full credit.

(Throughout this exam paper,
wff is an abreviation for *well formed formula(e)*)

1. (a) Construct truth-tables for $p \wedge q$, $p \vee q$, $p \to q$, $p \leftrightarrow q$ and $p\,\textbf{nand}\,q$ [5 marks].

   (b) Construct the parse-tree of $\neg(p \leftrightarrow q) \wedge (\neg p \to \neg r)$ [5 marks].

   (c) Construct the truth-table of $\neg(p \leftrightarrow q) \wedge (\neg p \to \neg r)$ [5 marks].

   (d) Construct a wff in disjunctive normal form that has the following truth-table [5 marks].

p	q	r	A
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	F
F	F	F	F

   (e) Prove that $\neg(p \vee (q \vee r))$ is logically equivalent to $\neg p \wedge (\neg q \wedge \neg r)$ using truth tables. Justify your answer [5 marks].

Exam continues ...

2. (a) Prove that **nand** is an adequate set of connectives on its own. You may assume that $\{\neg, \wedge\}$ is an adequate set of connectives. [5 marks].

   (b) Use **truth trees** to determine whether the following is satisfiable

$$(p \to q) \wedge (p \to \neg q)$$

   [5 marks].

   (c) Use **truth trees** to write the following in disjunctive normal form

$$p \to (q \to p)$$

   [5 marks].

   (d) Use **truth trees** to determine whether the following is a valid argument

$$p \vee q, \neg p \vee r \models q \vee r$$

   [5 marks].

   (e) Use **truth trees** to determine whether the following is a tautology

$$(p \vee (q \wedge \neg q)) \leftrightarrow p$$

   [5 marks].

Exam continues ...

3. (a) Prove that $a^2 = a$ in a Boolean algebra. To gain full credit, you must make explicit reference to the Boolean algebra axioms listed at the end of this exam paper [5 marks].

   (b) Write down the input/output table of a transistor [1 mark]. Prove that not-gates and or-gates can be constructed from transistors alone [4 marks].

   (c) Write down a Boolean expression for the following input/output behaviour [2 marks] and so construct a corresponding circuit using not-gates, and-gates and or-gates [3 marks].

x	y	z	u
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	0
0	0	0	1

Exam continues . . .

4. (a) Let $R(x)$ be the predicate 'x is a rabbit' and let $L(x)$ be the predicate 'x likes carrots'. Write down the first-order sentences 'all rabbits like carrots' and 'some rabbits like carrots' [5 marks].

   (b) Use **truth-trees** to prove that

$$(\exists x)[D(x) \to (\forall y)D(y)]$$

   is universally valid [10 marks].

Boolean algebra axioms

(B1) $(x + y) + z = x + (y + z)$.

(B2) $x + y = y + x$.

(B3) $x + 0 = x$.

(B4) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(B5) $x \cdot y = y \cdot x$.

(B6) $x \cdot 1 = x$.

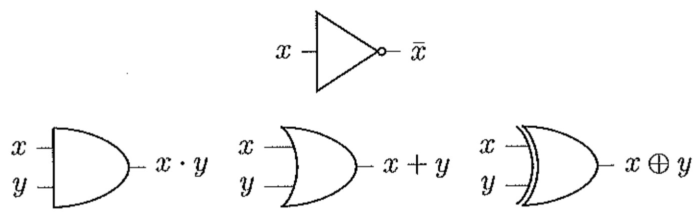
(B7) $x \cdot (y + z) = x \cdot y + x \cdot z$.

(B8) $x + (y \cdot z) = (x + y) \cdot (x + z)$.

(B9) $x + \bar{x} = 1$.

(B10) $x \cdot \bar{x} = 0$.

Circuit symbols



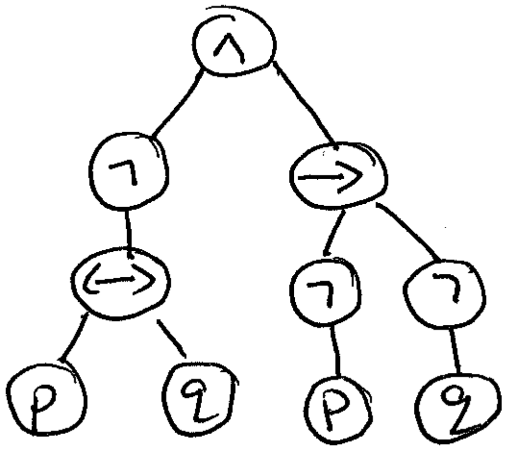
Exam ends

Logic and proof 2019 exam

1 (a).

p	q	$p \wedge q$ [1]	$p \vee q$ [1]	$p \rightarrow q$ [1]	p nand q [1]	$p \leftrightarrow q$ [1]
T	T	T	T	T	F	T
T	F	F	T	F	T	F
F	T	F	T	T	T	F
F	F	F	F	T	T	T

(b) [5]



[5]

(c)

p	q	r	$\neg(p \leftrightarrow q) \wedge (\neg p \rightarrow \neg r)$
T	F	T	T
T	F	F	T
F	T	F	T

[for brevity, I have just indicated where the outputs are T].

(d) $(p \wedge q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \equiv A$. [5]

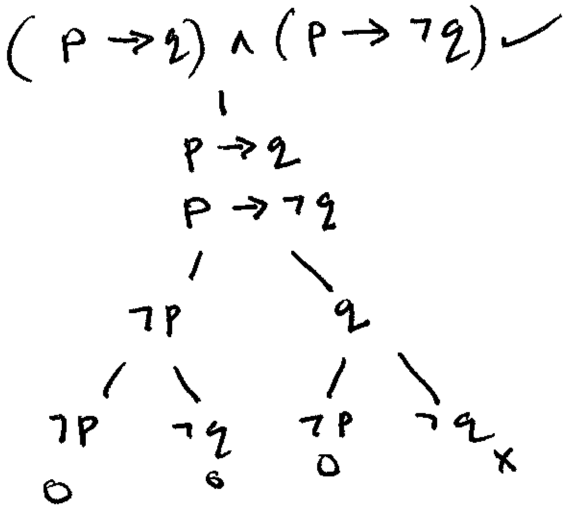
(e) $\neg(p \vee q \vee r)$ is only T when p, q, r are all F. $\neg p \wedge \neg q \wedge \neg r$ is only T when p, q, r are all F. Since the truth tables are the same, we can deduce that

$$\neg(p \vee q \vee r) \equiv \neg p \wedge \neg q \wedge \neg r.$$

2(a) $p \underline{\text{nand}}\, q \overset{\text{def}}{\equiv} \neg(p \wedge q)$

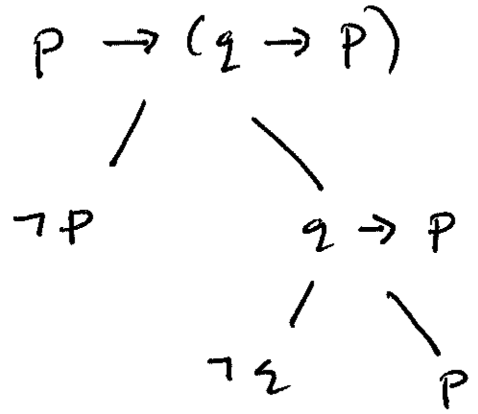
- $p \underline{\text{nand}}\, p = \neg(p \wedge p) \equiv \neg p$. [2].
- $p \wedge q \equiv \neg(\neg(p \wedge q))$
  $\equiv \neg(p \underline{\text{nand}}\, q)$
  $\equiv (p \underline{\text{nand}}\, q) \underline{\text{nand}} (p \underline{\text{nand}}\, q)$. [3]

(b)



satisfiable since there is an open branch. [5]

(c)



$\therefore \; p \to (q \to p) \equiv (\neg p) \vee (\neg q) \vee (p)$. [5]

(d) $p \vee q$ ✓

$\neg p \vee r$

NB $\neg(q \vee r)$ ✓

$\neg q$

$\neg r$

p | q ×

$\neg p$ | r

Tree closes and so argument is valid.

(e) NB $\neg[(p \vee (q \wedge \neg q)) \leftrightarrow p]$

$p \vee (q \wedge \neg q)$ | $\neg(p \vee (q \wedge \neg q))$

$\neg p$ | p

$\neg p$

$\neg(q \wedge \neg q)$

×

p × | $q \wedge \neg q$

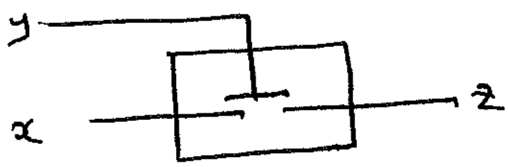
q

$\neg q$

×

Tree closes and so the original wff is a tautology.

3(a) $a = a \cdot 1$ by (B6)
$= a \cdot (a + \bar{a})$ by (B9)
$= a \cdot a + a\bar{a}$ by (B7)
$= a^2 + 0$ by (B10)
$= a^2$ by (B3) [5]

(b) 

(schematic diagram of a transistor)

x	y	z
1	1	0
1	0	1
0	1	0
0	0	0

[1]

$z = x \cdot \bar{y} = x \,\square\, y$ ← my notation

• $1 \,\square\, y = 1 \cdot \bar{y} = \bar{y}$. Thus we can construct a not-gate
[2]

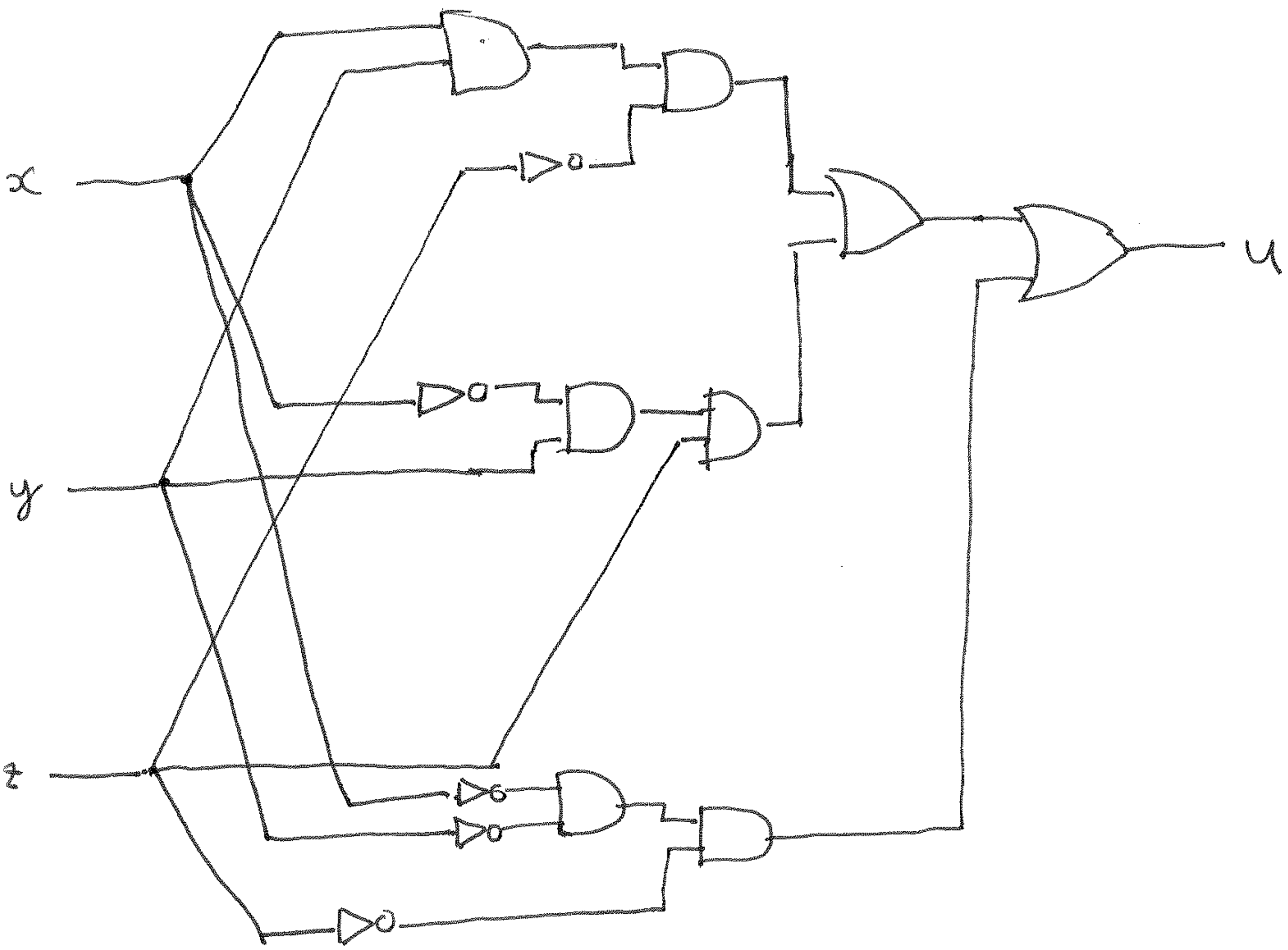
• $x + y = \overline{\overline{(x+y)}} = \overline{(\bar{x} \cdot \bar{y})} = \overline{(\bar{x} \,\square\, y)}$
$= \overline{((1 \,\square\, x) \,\square\, y)} = 1 \,\square\, ((1 \,\square\, x) \,\square\, y)$.

Thus we can construct an or-gate. [2].

3 (c)

$u = (x \cdot y \cdot \overline{z}) + (\overline{x} \cdot y \cdot z) + (\overline{x} \cdot \overline{y} \cdot \overline{z})$. [2]

[3]



4 (a)

'all rabbits like carrots' = $(\forall x)(R(x) \rightarrow L(x))$ [2½]

'some rabbits like carrots' = $(\exists x)(R(x) \wedge L(x))$ [2½]

$\neg(\exists x)[D(x) \rightarrow (\forall y)D(y)]$ ✓

NB

$(\forall x)\neg[D(x) \rightarrow (\forall y)D(y)]$ (*)

$\neg[D(a) \rightarrow (\forall y)D(y)]$ ✓

$D(a)$

$\neg(\forall y)D(y)$ ✓

$(\exists y)\neg D(y)$

$\neg D(b)$ ← new name

$\neg[D(b) \rightarrow (\forall y)D(y)]$

$D(b)$

$\neg(\forall y)D(y)$ [10]

X

Truth ~~taste~~ tree closes so wff is universally valid.