Boolean algebras

Axioms for a Boolean algebra

A *Boolean algebra* is defined by the following data $(B, +, \cdot, \bar{}, 0, 1)$ where B is a set that carries the structure, $+$ and $\cdot$ are binary operations, meaning that they have two, ordered inputs and one output, $a \mapsto \bar{a}$ is a unary operation, meaning that it has one input and one output, and two special elements of B: namely, 0 and 1. In addition, the following ten axioms are required to hold.

(B1): $(x + y) + z = x + (y + z)$.
(B2): $x + y = y + x$.
(B3): $x + 0 = x$.
(B4): $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
(B5): $x \cdot y = y \cdot x$.
(B6): $x \cdot 1 = x$.
(B7): $x \cdot (y + z) = x \cdot y + x \cdot z$.
(B8): $x + (y \cdot z) = (x + y) \cdot (x + z)$.
(B9): $x + \bar{x} = 1$.
(B10): $x \cdot \bar{x} = 0$.

These axioms are organized as follows. The first group of three (B1), (B2), (B3) deals with the properties of $+$ on its own: brackets, order, special element. The second group of three (B4), (B5), (B6) deals with the properties of $\cdot$ on its own: brackets, order, special element. The third group (B7), (B8) deals with how $+$ and $\cdot$ interact, with axiom (B8) being odd looking. The final group (B9), (B10) deals with the properties of $a \mapsto \bar{a}$, called *complementation*. On a point of notation, we shall usually write xy rather than $x \cdot y$.

Proposition *Let B be a Boolean algebra and let $a, b \in B$.*

(1) $a^2 = a \cdot a = a$. *Idempotence.*
(2) $a + a = a$. *Idempotence.*
(3) $a \cdot 0 = 0$. *Zero.*
(4) $1 + a = 1$. *Identity.*
(5) $a = a + a \cdot b$. *Absorption law.*
(6) $a + b = a + \bar{a} \cdot b$. *Absorption law.*

The 2-element Boolean algebra $\mathbb{B}$

The Boolean algebra we shall use in circuit design is the 2-element Boolean algebra $\mathbb{B}$. It is defined as follows. Put $\mathbb{B} = \{0, 1\}$. We define operations $\bar{}$, $\cdot$, and $+$ by means of the following tables. They are the same as the truth table for $\neg$, $\wedge$ and $\vee$ except that we replace T by 1 and F by 0.

x	$\bar{x}$
1	0
0	1

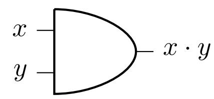
x	y	$x \cdot y$
1	1	1
1	0	0
0	1	0
0	0	0

x	y	$x + y$
1	1	1
1	0	1
0	1	1
0	0	0

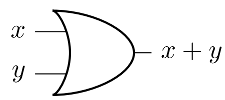
We shall mainly be working with expressions in x, y, z where the variables are *Boolean variables* meaning that $x, y, z \in \mathbb{B}$.

Gates

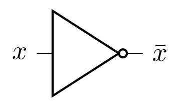
There are three basic gates that correspond to the three operations of a Boolean algebra. The *and-gate* is the function defined by $(x, y) \mapsto x \cdot y$. We use the following symbol to represent this function.



The *or-gate* is the function defined by $(x, y) \mapsto x + y$. We use the following symbol to represent this function.



Finally, the *not-gate* is the function defined by $x \mapsto \bar{x}$. We use the following symbol to represent this function.



Diagrams constructed using gates are called *circuits* and show how Boolean functions can be computed. *Such mathematical circuits can be converted into physical circuits with gates being constructed from simpler circuit elements called transistors.*

Example The input/output table below

x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

defines *exclusive or* or *xor*. We shall show that it can be constructed from and- or- and not-gates. But it is also convenient to have a special symbol for it.