HERIOT WATT UNIVERSITY

SCHOOL OF MATHEMATICAL AND COMPUTER SCIENCES

Department of Mathematics

F17LP

Logic and Proof

Semester 1 – 2018/19

Duration: 2 Hours

Attempt all questions

A University approved calculator may be used
for basic computations, but
appropriate working must be shown to obtain full credit.

+ SOLUTIONS

F17LP Logic and proof 2018

Each question is worth 20 marks

(Throughout this exam paper,
wff is an abreviation for *well formed formula(e)*)

1. (a) Construct truth-tables for $p \wedge q$, $p \vee q$, $p \to q$ and $p \leftrightarrow q$ [4 marks].
   (b) Construct the parse-tree of $(p \leftrightarrow q) \wedge (p \to \neg r)$ [2 marks].
   (c) Construct the truth-table of $(p \leftrightarrow q) \wedge (p \to \neg r)$ [4 marks].
   (d) Construct a wff in disjunctive normal form that has the following truth-table [4 marks].

p	q	r	A
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	T

   (e) Prove that $p \vee (q \wedge r)$ is logically equivalent to $(p \vee q) \wedge (p \vee r)$ using truth tables. Justify your answer [4 marks].
   (f) What is the *satisfiability problem*? Briefly explain its significance [2 marks].

Exam continues ...

2. (a) Define what is meant by *conjunctive normal form (CNF)* [1 mark]. Given the truth table for a wff A, explain, with reasons, how you would obtain a wff B, in CNF, which was logically equivalent to A [4 marks].

   (b) Define what is meant by a *Horn formula* [1 mark]. Write the following Horn formula, X, in *implicational form*

$$(p \vee \neg q \vee \neg r) \wedge (\neg s \vee \neg u) \wedge (\neg p \vee \neg q \vee r) \wedge (p) \wedge (q)$$

   [2 marks]. By using the **fast algorithm** (and showing all steps), determine whether X is satisfiable or not [2 marks].

   (c) Use **truth trees** to determine whether the following is a valid argument

$$a \to (b \to c), \neg d \vee a, b \models d \to c$$

   [5 marks].

   (d) Use **truth trees** to determine whether the following is a tautology

$$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$$

   [5 marks].

Exam continues ...

3. In this question, you should use the Boolean algebra axioms listed at the end of this exam paper. You should also assume that $a^2 = a$ and $a+a = a$ for all elements a of a Boolean algebra and De Morgan's laws.

   (a) Prove $a + 1 = 1$. To gain full credit you must make explicit reference to the Boolean algebra axioms [5 marks].

   (b) Write down the input/output table of a transistor [1 mark]. Prove that not-gates and and-gates can be constructed from transistors alone [4 marks].

   (c) Write down a Boolean expression for the following input/output behaviour and so construct a corresponding circuit using not-gates, and-gates and or-gates.

x	y	z	u
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	0

[10 marks].

Exam continues ...

4. (a) Use first-order logic to explain the two different meanings of the statement 'everyone loves someone' [10 marks].

   (b) Use **truth-trees** to prove that

$$(\forall x)(\forall y)R(x,y) \to (\forall x)R(x,x)$$

   is universally valid [10 marks].

Boolean algebra axioms

(B1) $(x+y)+z = x+(y+z)$.

(B2) $x+y = y+x$.

(B3) $x+0 = x$.

(B4) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(B5) $x \cdot y = y \cdot x$.

(B6) $x \cdot 1 = x$.

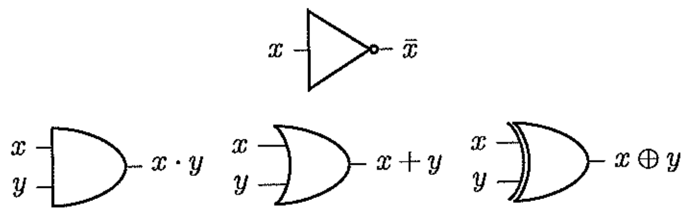
(B7) $x \cdot (y+z) = x \cdot y + x \cdot z$.

(B8) $x + (y \cdot z) = (x+y) \cdot (x+z)$.

(B9) $x + \bar{x} = 1$.

(B10) $x \cdot \bar{x} = 0$.

Circuit symbols



Exam ends

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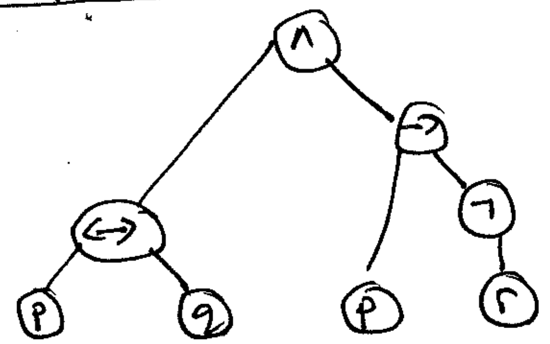
1 (a)

p	q	$p \wedge q$ [1]	$p \vee q$ [1]	$p \to q$ [1]	$p \leftrightarrow q$ [1]
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

(b)



[2]

(c)

[4]

p	q	r	$(p \leftrightarrow q) \wedge (p \to \neg r)$
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	T

(d) $(p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \wedge (\neg p \wedge q \wedge r) \wedge (\neg p \wedge \neg q \wedge \neg r)$

[4]

(e) $[1\frac{1}{2}]$ $[1\frac{1}{2}]$

p	q	r	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Truth tables are the same so $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$. [1]

(f) The satisfiability problem asks whether a wff is satisfiable or not. It is NP-Complete. [1+1]

$(\vee ..) \wedge (\vee ..) \wedge \ldots$

2 (a) A conjunction of blocks each block being a ~~[illegible]~~ disjunction of literals. [1]

- Construct the truth table of $\neg A$.
- Write $\neg A$ in DNF. ie $\neg A \equiv B$.
- Negate both sides of $\neg A \equiv B$ to get $A \equiv \neg B$.
- Pushing $\neg$ through B and using double negation we get a wff in CNF. [4]

(b) It is a wff in CNF in which each block has at most one positive literal. [1]

$$(q \wedge r \to p) \wedge (s \wedge u \to \underline{f}) \wedge (p \wedge q \to r) \wedge (\underline{t} \to p) \wedge (\underline{t} \to q)$$

[2]

Now apply the [illegible] algorithm

$$(\dot{q} \wedge \dot{r} \to \dot{p}) \wedge (s \wedge u \to \underline{f}) \wedge (\dot{p} \wedge \dot{q} \to \dot{r}) \wedge (\underline{t} \to \dot{p}) \wedge (\underline{t} \to \dot{q})$$

[2]

p	q	r	s		u
T	T	T	F		F

It is satisfiable.

2(c).

$a \to (b \to c)$

$\neg d \vee a$ ✓

b

NB $\neg(d \to c)$ ✓

d

$\neg c$

$\neg d$ × | a

$\neg a$ × | $b \to c$

$\neg b$ × | c ×

[1]

Because the tree has no open branches the argument is valid.

[4]

(d)

NB $\neg[(p \to (q \to r)) \to ((p \to q) \to (p \to r))]$ ✓

$p \to (q \to r)$

$\neg[(p \to q) \to (p \to r)]$ ✓

[4]

$p \to q$ ✓

$\neg(p \to r)$ ✓

p

$\neg r$

$\neg p$ × | q

$\neg p$ × | $q \to r$

$\neg q$ × | r ×

[1]

Because the tree has no open branches the wff is a tautology.

3

(a) $a + 1 = a + (a + \bar{a})$ by (B9) [5]

$= (a + a) + \bar{a}$ by (B1)

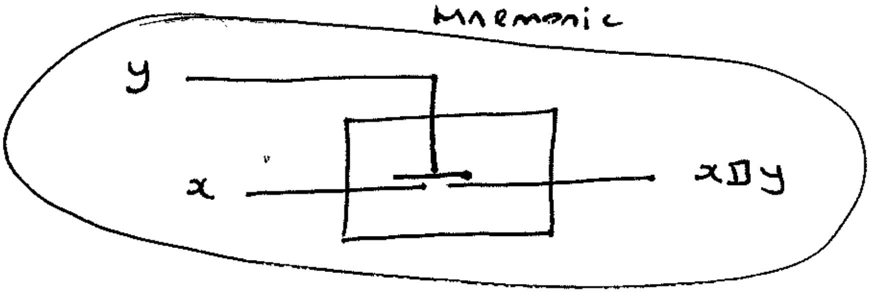
$= a + \bar{a}$ since $a + a = a$

$= 1$ by (B9)

[1]

(b)

x	y	$x \square y$
1	1	0
1	0	1
0	1	0
0	0	0

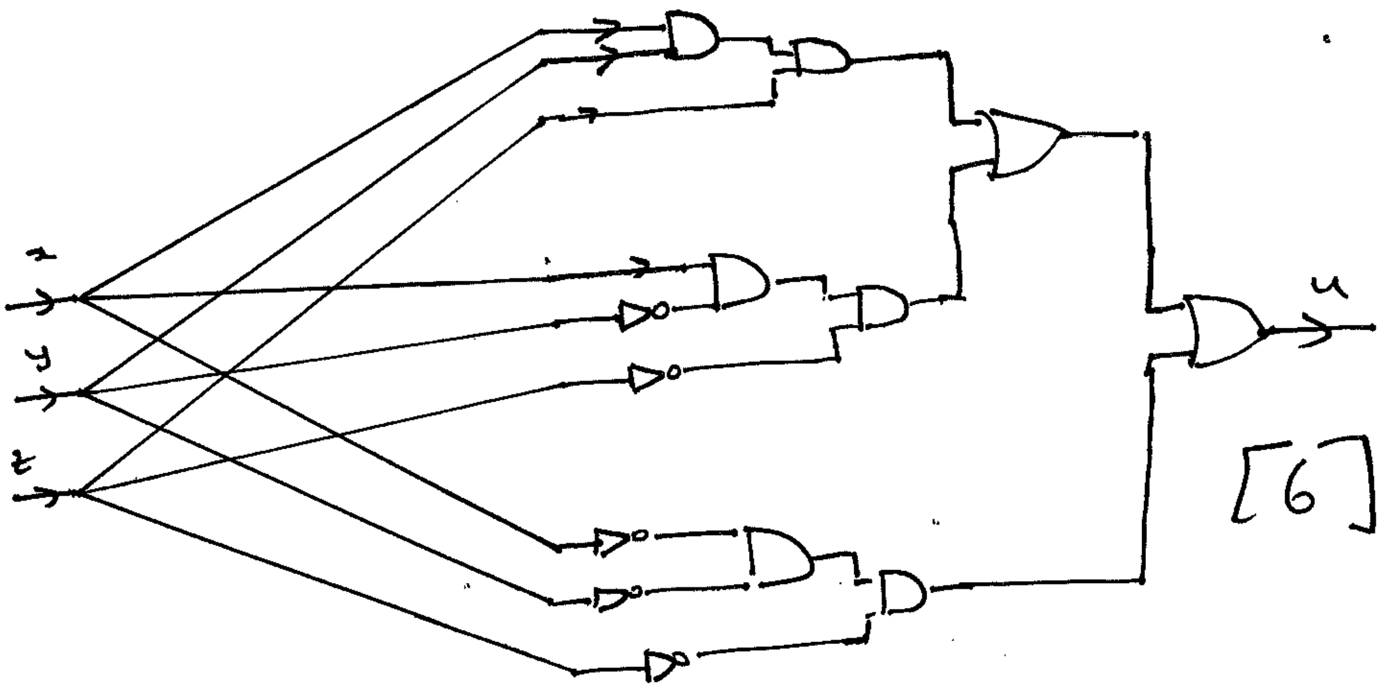


So, $x \square y = x \cdot \bar{y}$

- $1 \square y = 1 \cdot \bar{y} = \bar{y}$. This gives not-gates. [2]
- [2]

$x \cdot y = x \cdot \bar{\bar{y}} = x \cdot \overline{(\bar{y})} = x \square \bar{y} = x \square (1 \square y)$

(c) The Boolean expression describing the circuit is $u = x.y.z + x.\bar{y}.\bar{z} + \bar{x}.\bar{y}.\bar{z}$. [4]



[6]

4.(a) $L(x,y)$ = 'x loves y'

$(\forall x)(\exists y)\, L(x,y)$. This means that for each person there is someone whom they love [5]

$(\exists y)(\forall x)\, L(x,y)$. This means that there is someone whom everyone loves. [5]

(b)

$$\neg[(\forall x)(\forall y)R(x,y) \rightarrow (\forall x)R(x,x)] \checkmark$$

NB

$$(\forall x)(\forall y)R(x,y) \ *$$
$$\neg(\forall x)R(x,x) \checkmark$$

[10]

$$(\exists x)\neg R(x,x) \checkmark$$
$$\neg R(a,a)$$
$$(\forall y)R(a,y) \ *$$
$$R(a,a)$$
$$\times$$