Lecture 11

1.8 Normal forms

I could write $\neg\neg\neg\neg\neg p$ instead of $\neg p$. This would be allowable but hardly 'normal'. In this section, I describe some standard ways of writing wff that will lead to a connection with PROLOG.

Negation normal form (NNF)

Definition A wff is in negation normal form NNF if it is constructed using only $\vee$, $\wedge$ and literals (Recall that a literal is either an atom or the negation of an atom).

<u>Proposition</u> Every wff is logically equivalent to a wff in NNF.

<u>Proof</u> Replace $p \oplus q$ by $\neg(p \leftrightarrow q)$

— $p \leftrightarrow q$ by $(p \to q) \wedge (q \to p)$

— $p \to q$ by $\neg p \vee q$

Thus, as we saw earlier, every wff is logically equivalent to one using only $\neg, \wedge, \vee$. <u>BUT</u> this is not enough to get NNF because any ~~negatives~~ negations <u>must</u> occur immediately in front of atoms. To do this, we use De Morgan's laws and double negation. ∎

<u>Example</u> We show that $\neg(p \to (p \wedge q))$ can be written in NNF.

not in NNF

$$\neg(p \to (p \wedge q)) \equiv \neg(\neg p \vee (p \wedge q))$$

$$\equiv \neg\neg p \wedge \neg(p \wedge q)$$

$$\equiv \neg\neg p \wedge (\neg p \vee \neg q)$$

$$\equiv \underline{\underline{p \wedge (\underline{\neg p} \vee \underline{\neg q})}}$$

NNF.

<u>Example</u> Write $\neg(p \leftrightarrow (q \to r))$ in NNF.

$$\neg(p \leftrightarrow (q \to r)) \equiv \neg[(p \to (q \to r)) \wedge ((q \to r) \to p)]$$

$$\equiv \neg[(\neg p \vee (\neg q \vee r)) \wedge (\neg(\neg q \vee r) \vee p)]$$

$$\equiv \neg[(\neg p \vee \neg q \vee r) \wedge ((\neg\neg q \wedge \neg r) \vee p)]$$

$$\equiv (\neg\neg p \wedge \neg\neg q \wedge \neg r) \vee (\neg(q \wedge \neg r) \wedge \neg p)$$

$$\equiv \underbrace{(p \wedge q \wedge \neg r) \vee ((\neg q \vee r) \wedge \neg p)}_{\text{NNF}}$$

p	q	r	$\neg(p\leftrightarrow(q\rightarrow r))$
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	T

p	q	r	$(p\wedge(q\wedge\neg r))\vee((\neg q\vee r)\wedge\neg p)$
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	T

As expected, the two truth tables are the same (because the wffs are logically equivalent).

Disjunctive normal form (DNF)

A wff is in DNF if it has the following shape

Disjunctions outside

$$(\wedge \text{ literals}) \vee (\wedge \text{ literals}) \vee \dots \vee (\wedge \text{ literals})$$

might only have one bracket!

Extreme examples

(1) p is in DNF (p)

(2) $p \vee q$ is in DNF $(p) \vee (q)$

(3) $p \wedge q$ is in DNF $(p \wedge q)$

Theorem Every wff is logically equivalent to one in DNF.

The easiest way to see this is to use our method for constructing up from truth functions.

<u>Example</u> The truth table for

$\neg(p \leftrightarrow (q \rightarrow r)) = A$

p	q	r	A
T	T	T	F
T	T	F	T *
T	F	T	F
T	F	F	F
F	T	T	T *
F	T	F	F
F	F	T	T *
F	F	F	T *

$$A \equiv (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r)$$
$$\vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r) \leftarrow \text{DNF}$$

However, we can also use the calculations we did earlier. We proved that

$$A \equiv (p \wedge q \wedge \neg r) \vee \Big(\underbrace{(\neg q \vee r) \wedge \neg p}_{\downarrow \text{ distributivity}} \Big)$$

$$\equiv (p \wedge q \wedge \neg r) \vee \Big((\neg q \wedge \neg p) \vee (r \wedge \neg p) \Big)$$

$$\equiv (p \wedge q \wedge \neg r) \vee (\neg q \wedge \neg p) \vee (r \wedge \neg p)$$

DNF

[DNF is not unique]

Conjunctive normal form (CNF)

$(\bigvee \text{literals}) \wedge (\bigvee \text{literals}) \wedge \ldots$

<u>Proposition</u> Every wff is logically equivalent to one in CNF.

<u>Proof</u> Write $\neg A$ in DNF and then negate both sides. $\blacksquare$

Example The truth table for A is

p	q	r	A
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	T

p	q	r	$\neg A$
T	T	T	F
T	T	F	F
T	F	T	T *
T	F	F	F
F	T	T	F
F	T	F	T *
F	F	T	F
F	F	F	F

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$\neg A \equiv (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r)$

$\neg\neg A \equiv \neg(p \wedge q \wedge \neg r) \wedge \neg(\neg p \wedge q \wedge \neg r)$

$A \equiv (\neg p \vee \neg q \vee r) \wedge (p \vee \neg q \vee r)$ CNF