Lecture 23

Boolean algebra A set B with operations $+$, $\cdot$, $-$ and constants $0, 1$ satisfying the following axioms:

(B1)	$(x+y)+z = x+(y+z)$.	(B4)	$(x\cdot y)\cdot z = x\cdot(y\cdot z)$.
(B2)	$x+y = y+x$.	(B5)	$x\cdot y = y\cdot x$.
(B3)	$x+0 = x$	(B6)	$x\cdot 1 = x$

(B7) $x\cdot(y+z) = x\cdot y + x\cdot z$.

(B8) $x+(y\cdot z) = (x+y)\cdot(x+z)$

(B9) $x+\bar{x} = 1$ (B10) $x\cdot\bar{x} = 0$

We usually write ab instead of $a\cdot b$

a^2 means aa.

We now make some important deductions from these axioms. Look carefully at the format of the proofs

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<u>Result</u> $a^2 = a$.

keystep

<u>Proof</u>

$$\boxed{a = a1} \qquad (B6)$$
$$= a(a+\bar{a}) \qquad (B9)$$
$$= a^2 + a\bar{a} \qquad (B7)$$
$$= a^2 + 0 \qquad (B10)$$
$$= a^2 \qquad (B3)$$

<u>Result</u> $a + a = a$

<u>Proof</u> similar to the one above

Result $a0 = 0$.

keystep

Proof $\boxed{a0 = a(a\bar{a})}$ (B10)

$= a^2\bar{a}$ (B4)

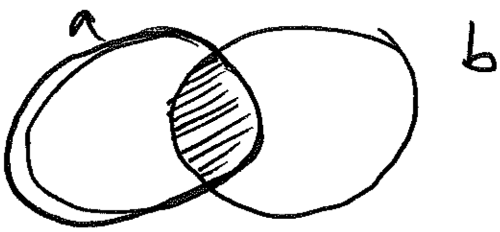
$= a\bar{a}$ (B10)

$= 0$

Result $a + 1 = 1$

Proof similar to one done

Result (Absorption law)



$a = a + ab$

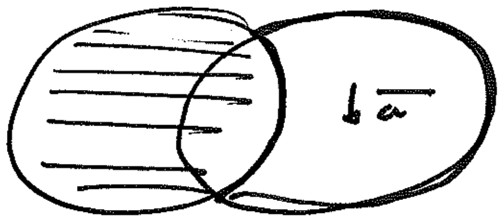
Proof $a + ab = \underline{a1} + ab$ (B6)

$= a(1+b)$ (B7)

$= a1$ $\quad 1 = 1+b$ proved earlier

$= a$ (B6)

Result (Absorption law) $a + b = a + \bar{a} b$



Proof

$$\begin{aligned} a + b &= a + \underline{\underline{1}}\, b \\ &= a + (a + \bar{a}) b \\ &= \underbrace{a + ab}{} + \bar{a} b \\ &= a + \bar{a} b \end{aligned}$$

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see exercises

Theorem (Much harder) De Morgan for BAs

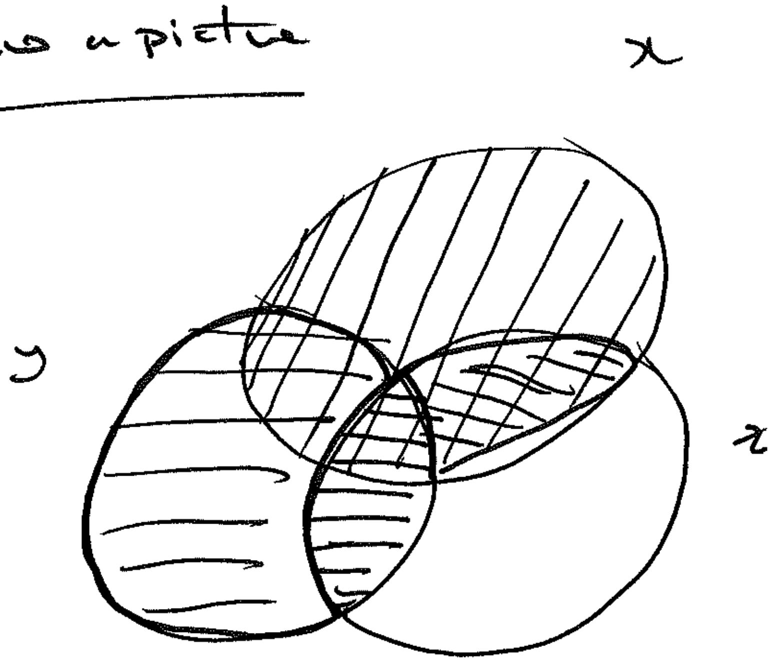
(1) $\overline{(a+b)} = \bar{a}\bar{b}$

(2) $\overline{(ab)} = \bar{a} + \bar{b}$

Example These results can be used to simplify Boolean expression. This is needed in designing circuits.

Simplify $x + yz + y\overline{x} + xz\overline{y}$

Draw a picture



Looks like $x + y$ from the Venn diagram.

We now prove this using BA

$$x + yz + y\bar{x} + xz\bar{y}$$

$$= \boxed{x + y\bar{x}} + yz + xz\bar{y} \qquad \text{assoc + comm}$$

$$= (x + y) + yz + xz\bar{y} \qquad \text{by absorption.}$$

$$= x + (y + yz) + xz\bar{y}$$

$$= x + y + xz\bar{y} \qquad \text{by absorption}$$

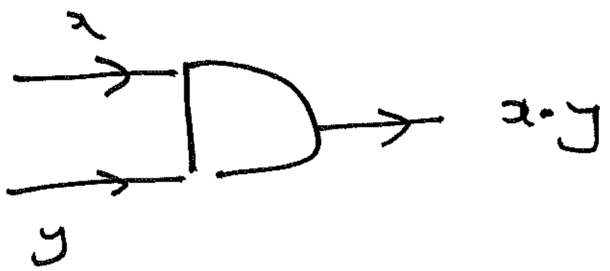
$$= x + x(z\bar{y}) + y$$

$$= x + y \qquad \text{by absorption.}$$

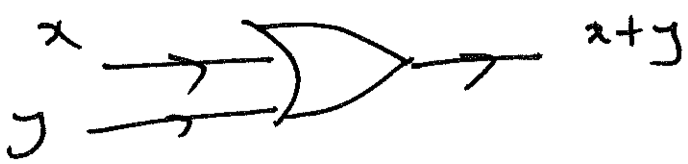
Circuit design

We shall work with the 2-element BA $\mathbb{B} = \{0, 1\}$.

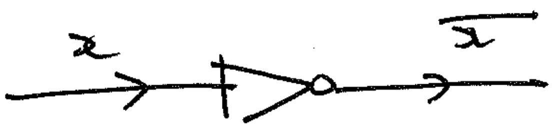
There are circuits that carry out the basic Boolean operations



and-gate



or-gate



not-gate

A *gate* is a circuit element that implements a Boolean operation.