Lecture 7

Section 1.4

Logical equivalence

__Example__ The wff $\neg\neg p$ and p are different so $\neg\neg p \neq p$. But they do have the same truth tables (they mean the same things).

p	$\neg\neg p$
T	T
F	F

p	p
T	T
F	F

We write $\neg\neg p \equiv p$ and say they are __logically equivalent__.

Examples

(1) $p \to q$ and $\neg p \vee q$ have the same truth tables.

(2) $p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$ have the same truth tables.

(3) $p \oplus q$ and $(p \vee q) \wedge \neg(p \wedge q)$ have the same truth tables.

Definition. If A and B are wff and have the same truth tables, we say that A and B are logically equivalent and write $A \equiv B$.

Remark $\equiv$ is not a logical connective.

It is a relation ~~between~~ wffs.

Think of it as 'generalized equality'!

<u>Example</u> Using this notation, we have that:

(1) $p \to q \equiv \neg p \vee q$.

(2) $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$.

(3) $p \oplus q \equiv (p \vee q) \wedge \neg (p \wedge q)$.

However (be careful!), A and B do not have to have the same atoms if $A \equiv B$.

<u>Example</u> $p \equiv p \wedge (q \vee \neg q)$

In order to compare the truth table for p with the truth table for $p \wedge (q \vee \neg q)$ we have to pad out p's truth table with a <u>dummy</u> <u>q</u> (even though this plays no rôle).

p	q	p
T	T	T
T	F	T
F	T	F
F	F	F

~~$p \wedge (q \vee \neg$~~.

p	q	$p \wedge (q \vee \neg q)$
T	T	T
T	F	T
F	T	F
F	F	F

truth tables the same

This is a bit artificial (but ok). Another way to show that $A \equiv B$ is to use the following result. This also begins to explain why tautologies are so important in PL.

Theorem Let A & B be wffs.
Then $A \equiv B$ if & only if $\models A \Leftrightarrow B$.

Proof. There are two statements we have to prove:

(1) If $A \equiv B$ then $\models A \Leftrightarrow B$.

(2) If $\models A \Leftrightarrow B$ then $A \equiv B$.

Proof of (1) Suppose that $A \equiv B$ (so, A & B have the same truth tables – possibly padded).
If $A \Leftrightarrow B$ is not a tautology, then there must be ~~a~~ an assignment of truth values to the atoms making A & B have different truth values. But this is impossible since A and B have the same truth tables. ■
It follows that $\models A \Leftrightarrow B$

Proof of (2) Suppose that $\models A \Leftrightarrow B$.
If $A \not\equiv B$ then there is therefore an assignment of truth values to the atoms making A & B have different truth values. But then $A \Leftrightarrow B$ cannot be a tautology. It follows that
$A \equiv B$. ■

<u>Examples</u> Construct truth tables of the following

(1) $(p \to q) \leftrightarrow (\neg p \vee q)$.

(2) $p \leftrightarrow (p \wedge (q \vee \neg q))$.

(3) $\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$.

p	q	(p→q)↔(¬pVq)
T	T	T
T	F	T
F	T	T
F	F	T

This is a tautology and so

$p \to q \equiv \neg p \vee q$

p	q	p↔(pΛ(qV¬q))
T	T	T
T	F	T
F	T	T
F	F	T

This is a tautology and so

$p \equiv p \wedge (q \vee \neg q)$

p	q	¬(pΛq)↔(¬pV¬q)
T	T	T
T	F	T
F	T	T
F	F	T

This is a tautology and so

$\neg(p \wedge q) \equiv \neg p \vee \neg q$