# Predicate Logic 2

## Semantics and Validity

### 2–1. INTERPRETATIONS

Recall that we used truth tables to give very precise definitions of the meaning of ‘&’, ‘v’ ‘~’, ‘⊃’, and ‘≡’. We would like to do the same for the meaning of quantifiers. But, as you will see very soon, truth tables won’t do the job. We need something more complicated.

When we were doing sentence logic, our atomic sentences were just sentence letters. By specifying truth values for all the sentence letters with which we started, we already fixed the truth values of **any** sentence which we could build up from these smallest pieces. Now that we are doing predicate logic, things are not so easy. Suppose we are thinking about all the sentences which we could build up using the one place predicate ‘B’, the two place predicate ‘L’, the name ‘a’, and the name ‘e’. We can form six atomic sentences from these ingredients: ‘Ba’, ‘Be’, ‘Laa’, ‘Lae’, ‘Lea’, and ‘Lee’. The truth table formed with these six atomic sentences would have 64 lines. Neither you nor I are going to write out a 64-line truth table, so let’s consider just one quite typical line from the truth table:

| Ba, | Be, | Laa, | Lae, | Lea, | Lee |
|---|---|---|---|---|---|
| t | f | f | t | f | t |

$\boxed{\dot{a} \quad \dot{e}}$

Figure 2–1

Even such an elementary case in predicate logic begins to get quite complicated, so I have introduced a pictorial device to help in thinking about such cases (see figure 2–1). I have drawn a box with two dots inside, one labeled 'a' and the other labeled 'e'. This box is very different from a Venn diagram. This box is supposed to picture just one way the whole world might be. In this very simple picture of the world, there are just two things, Adam and Eve. The line of the truth table on the left gives you a completed description of what is true and what is false about Adam and Eve in this very simple world: Adam is blond, Eve is not blond, Adam does not love himself, Adam does love Eve, Eve does not love Adam, and Eve does love herself.

You can also think of the box and the description on the left as a very short novel. The box gives you the list of characters, and the truth table line on the left tells you what happens in this novel. Of course, the novel is not true. But if the novel were true, if it described the whole world, we would have a simple world with just Adam and Eve having the properties and relations described on the left.

Now, in writing this novel, I only specified the truth value for atomic sentences formed from the one and two place predicates and from the two names. What about the truth value of more complicated sentences? We can use our old rules for figuring out the truth value of compounds formed from these atomic sentences using '&', 'v', '~', '⊃', and '≡'. For example, in this novel 'Ba & Lae' is true because both the components are true.

What about the truth value of '(∃x)Bx'? Intuitively, '(∃x)Bx' should be true in the novel because in the novel there is someone, namely Adam, who is blond. As another example, consider '(∃x)Lxa'. In this novel '(∃x)Lxa' is false because Eve does not love Adam and Adam does not love Adam. And in this novel there isn't anyone (or anything) else. So no one loves Adam. In other words, in this novel it is false that there is someone who loves Adam.

Let's move on and consider the sentence '(∀x)Lxe'. In our novel this sentence is true, because Adam loves Eve, and Eve loves herself, and that's all the people there are in this novel. If this novel were true, it would be true that everyone loves Eve. Finally, '(∀x)Bx' is false in the novel, for in this novel Eve is not blond. So in this novel it is false that everyone is blond.

Remember what we had set out to do: We wanted to give a precise account of the meaning of the quantifiers very like the precise account which truth table definitions gave to '&' and the other sentence logic connectives. In sentence logic we did this by giving precise rules which told us when a compound sentence is true, given the truth value of the compound's components.

We now have really done the same thing for '(∀x)' and '(∃x)' in one special case. For a line of a truth table (a "novel") that gives a truth value for all atomic sentences using 'B', 'L', 'a', and 'e', we can say whether a universally quantified or an existentially quantified sentence is true or false. For example, the universally quantified sentence '(∀x)Lxe' is true just in case 'Lxe' is true for all values of 'x' in the novel. At the moment we are considering a novel in which the only existing things are Adam and Eve. In such a novel '(∀x)Lxe' is true if **both** 'Lxe' is true when we take 'x' to refer to Adam **and** 'Lxe' is also true when we take 'x' to refer to Eve. Similarly, '(∃x)Bx' is true in such a novel just in case 'Bx' is true for some value of 'x' in the novel. As long as we continue to restrict attention to a novel with only Adam and Eve as characters, '(∃x)Bx' is true in the novel if **either** 'Bx' is true when we take 'x' to refer to Adam **or** 'Bx' is true if we take 'x' to refer to Eve.



If the example seems a bit complicated, try to focus on this thought: All we are really doing is following the intuitive meaning of "all x" and "some x" in application to our little example. If you got lost in the previous paragraph, go back over it with this thought in mind.

Now comes a new twist, which might not seem very significant, but which will make predicate logic more interesting (and much more complicated) than sentence logic. In sentence logic we always had truth tables with a finite number of lines. Starting with a fixed stock of atomic sentence letters, we could always, at least in principle, write out all possible cases to consider, all possible assignments of truth values to sentence letters. The list might be too long to write out in practice, but we could at least understand everything in terms of such a finite list of cases.

Can we do the same thing when we build up sentences with predicates and names? If, for example, we start with just 'B', 'L', 'a', and 'e', we can form six atomic sentences. We can write out a 64-line truth table which will give us the truth value for any compound built up from these six atomic sentences, for any assignment of truth values to the atomic sentences. But the fact that we are using quantifiers means that we must also consider further possibilities.

Consider the sentence '(∀x)Bx'. We know this is false in the one case we used as an example (in which 'Ba' is true and 'Be' is false). You will immediately think of three alternative cases (three alternative "novels") which must be added to our list of relevant possible cases: the case in which Eve is blond and Adam is not, the case in which Adam and Eve are both blond, and the case in which both are not blond. But there are still more cases which we must include in our list of all possible cases! I can generate more cases by writing new novels with more characters. Suppose I write a new novel with Adam, Eve, and Cid. I now have eight possible ways of distributing hair color (blond or not blond) among my characters, which can be combined with 512 different possible combinations of who does or does not love whom! And, of course, this is just the beginning of an unending list of novels describing possible cases in which '(∀x)Bx' will have a truth value. I can always expand my list of novels by adding new characters. I can even describe novels with infinitely many characters, although I would not be able to write such a novel down.

How are we going to manage all this? In sentence logic we always had, for a given list of atomic sentence, a finite list of possible cases, the finite number of lines of the corresponding truth table. Now we have infinitely many possible cases. We can't list them all, but we can still say what any one of these possible cases looks like. Logicians call a possible case for a sentence of predicate logic an *Interpretation* of the sentence. The example with which we started this chapter is an example of an interpretation, so actually you have already seen and understood an example of an interpretation. We need only say more generally what interpretations are.

We give an interpretation, first, by specifying a collection of objects which the interpretation will be about, called the *Domain* of the interpretation. A domain always has at least one object. Then we give names to the objects in the domain, to help us in talking about them. Next, we must say which predicates will be involved. Finally, we must go through the predicates and objects and say which predicates are true of which objects. If we are concerned with a one place predicate, the interpretation specifies a list of objects of which the object is true. If the predicate is a two place predicate, then the interpretation specifies a list of **pairs** of objects between which the two place relation is supposed to hold, that is, pairs of objects of which the two place relation is true. Of course, order is important. The pair a-followed-by-b counts as a different pair from the pair b-followed-by-a. Also, we must consider objects paired with themselves. For example, we must specify whether Adam loves himself or does not love himself. The interpretation deals similarly with three and more place predicates.

In practice, we often specify the domain of an interpretation simply by giving the interpretation's names for those objects. I should mention that in a fully developed predicate logic, logicians consider interpretations which have unnamed objects. In more advanced work, interpretations of this kind become very important. But domains with unnamed objects would make it more difficult to introduce basic ideas and would gain us nothing for the work we will do in part I of this volume. So we won't consider interpretations with unnamed objects until part II.

The following gives a summary and formal definition of an interpretation:

An *Interpretation* consists of

a) A collection of objects, called the interpretation's *Domain*. The domain always has at least one object.
b) A name for each object in the domain. An object may have just one name or more than one name. (In part II we will expand the definition to allow domains with unnamed objects.)



c) A list of predicates.
d) A specification of the objects of which each predicate is true and the objects of which each predicate is false—that is, which one place predicates apply to which individual objects, which two place predicates apply to which pairs of objects, and so on. In this way every atomic sentence formed from predicates and names gets a truth value.
e) An interpretation may also include atomic sentence letters. The interpretation specifies a truth value for any included atomic sentence letter.

By an *Interpretation of a Sentence*, we mean an interpretation which is sure to have enough information to determine whether or not the sentence is true or false in the interpretation:

An *Interpretation of a Sentence* is an interpretation which includes all the names and predicates which occur in the sentence and includes truth values for any atomic sentence letters which occur in the sentence.

For example, the interpretation of figure 2–1 is an interpretation of 'Ba' and of '(∀x)Lxx'. In this interpretation 'Ba' is true and '(∀x)Lxx' is false. Note that for each of these sentences, the interpretation contains more information than is needed to determine whether the sentence is true or false. This same interpretation is not an interpretation of 'Bc' or of '(∃x)Txe'. This is because the interpretation does not include the name 'c' or the two place predicate 'T', and so can't tell us whether sentences which use these terms are true or false.

### EXERCISES

2–1. I am going to ask you to give an interpretation for some sentences. You should use the following format. Suppose you are describing an interpretation with a domain of three objects named 'a', 'b', and 'c'. Specify the domain in this way: D = {a,b,c}. That is, specify the domain by giving a list of the names of the objects in the domain. Then specify what is true about the objects in the domain by using a sentence of predicate logic. Simply conjoin all the atomic and negated atomic sentences which say which predicates are true of which objects and which are false. Here is an example. The following is an interpretation of the sentence 'Tb & Kbd':

D = {b,d}; Tb & Td & Kbb & Kbd & Kdb & Kdd.

In this interpretation all objects have property T and everything stands in the relation K to itself and to everything else. Here is another interpretation of the same sentence:

D = {b,d}; ~Tb & Td & Kbb & ~Kbd & ~Kdb & Kdd.

Sometimes students have trouble understanding what I want in this exercise. They ask, How am I supposed to decide which interpretation to write down? You can write down any interpretation you want as long as it is **an** interpretation of the sentence I give you. In every case you have infinitely many interpretations to choose from because you can always get more interpretations by throwing in more objects and then saying what is true for the new objects. Choose any you like. Just make sure you are writing down an interpretation of the sentence I give you.

a) Lab
b) Lab ⊃ Ta
c) Lab v ~Lba
d) (∀x)(Fx≡Rxb)
e) Ga & (∃x)(Lxb v Rax)
f) (Kx & (∀x)Rax) ⊃ (∃x)(Mx v Rcx)

## 2–2. TRUTH IN AN INTERPRETATION

Just like a line of a truth table, an interpretation tells us whether each atomic sentence formed from predicates and names is true or false. What about compound sentences? If the main connective of a compound sentence does not involve a quantifier, we simply use the old rules for the connectives of sentence logic. We have only one more piece of work to complete: We must make more exact our informal description of the conditions under which a quantified sentence is true or is false in an interpretation.

Intuitively, a universally quantified sentence is going to be true in an interpretation if it is true in the interpretation for **every**thing to which the variable could refer in the interpretation. (Logicians say, "For **every** value of the universally quantified variable.") An existentially quantified sentence will be true in an interpretation if it is true for **some**thing to which the variable could refer in the interpretation (that is, "for **some** value of the existentially quantified variable.") What we still need to do is to make precise what it is for a quantified sentence to be true for a value of a variable. Let's illustrate with the same example we have been using, the interpretation given in figure 2–1.

Consider the sentence '(∀x)Bx'. In the interpretation we are considering, there are exactly two objects, a, and e. '(∀x)Bx' will be true in the interpretation just in case, roughly speaking, it is true both for the case of 'x' referring to a and the case of 'x' referring to e. But when 'x' refers to a, we have the sentence 'Ba'. And when 'x' refers to 'e', we have the sentence 'Be'. Thus '(∀x)Bx' is true in this interpretation just in case both 'Ba' and 'Be' are true. We call 'Ba' the *Substitution Instance* of '(∀x)Bx' formed by substituting 'a' for 'x'. Likewise, we call 'Be' the substitution instance of '(∀x)Bx' formed by substituting 'e' for 'x'. Our strategy is to explain the meaning of universal quantification by defining this notion of substitution instance and then specifying that a universally quantified sentence is true in an interpretation just in case it is true for all substitution instances in the interpretation:



> (Incomplete Definition) For any universally quantified sentence (∀**u**)(. . . **u** . . .), the *Substitution Instance* of the sentence with the name **s** substituted for the variable **u** is (. . . **s** . . .), the sentence formed by dropping the initial universal quantifier and writing **s** wherever **u** had occurred.

A word of warning: This definition is not yet quite right. It works only as long as we don't have multiple quantification, that is, as long as we don't have sentences which stack one quantifier on top of other quantifiers. But until chapter 3 we are going to keep things simple and consider only simple sentences which do not have one quantifier applying to a sentence with another quantifier inside. When we have the basic concepts we will come back and give a definition which is completely general.

Now we can easily use this definition of substitution instance to characterize truth of a universally quantified sentence in an interpretation:

> (Incomplete Definition) *A universally quantified sentence is true in an interpretation* just in case **all** of the sentence's substitution instances, formed with names in the interpretation, are true in the interpretation.

Another word of warning: As with the definition of substitution instance, this definition is not quite right. Again, chapter 3 will straighten out the details.

To practice, let's see whether '(∀x)(Bx ⊃ Lxe)' is true in the interpretation of figure 2–1. First we form the substitution instances with the names of the interpretation, 'a', and 'e'. We get the first substitution instance by dropping the quantifier and writing in 'a' everywhere we see 'x'. This gives

Ba ⊃ Lae.

Note that because 'Ba' and 'Lae' are both true in the interpretation, this first substitution instance is true in the interpretation. Next we form the second substitution instance by dropping the quantifier and writing in 'e' wherever we see 'x':

Be ⊃ Lee.

Because 'Be' is false and 'Lee' is true in the interpretation, the conditional 'Be ⊃ Lee' is true in the interpretation. We see that all the substitution instances of '(∀x)(Bx ⊃ Lxe)' are true in the interpretation. So this universally quantified sentence is true in the interpretation.

To illustrate further our condition for truth of a universally quantified sentence, consider the sentence '(∀x)(Bx ⊃ Lxa)'. This has the substitution instance 'Ba ⊃ Laa'. In this interpretation 'Ba' is true and 'Laa' is false, so 'Ba ⊃ Laa' is false in the interpretation. Because '(∀x)(Bx ⊃ Lxa)' has a false substitution instance in the interpretation, it is false in the interpretation.

You may have noticed the following fact about the truth of a universally quantified sentence and the truth of its substitution instances. By definition '(∀x)(Bx ⊃ Lxe)' is true in the interpretation just in case all of its instances are true in the interpretation. But its instances are all true just in case the **conjunction** of the instances is true. That is, '(∀x)(Bx ⊃ Lxe)' is true in the interpretation just in case the conjunction

(Ba ⊃ Lae) & (Be ⊃ Lee)

is true in the interpretation. If you think about it, you will see that this will hold in general. In the interpretation we have been discussing (or any interpretation with two objects named 'a' and 'e'), any universally quantified sentence, '(∀x)(. . . x . . .)', will be true just in case the conjunction of its substitution instance, '(. . . a . . .)&(. . . e . . .)', is true in the interpretation.

It's looking like we can make conjunctions do the same work that the universal quantifier does. A universally quantified sentence is true in an interpretation just in case the conjunction of all its substitution instances is true in the interpretation. Why, then, do we need the universal quantifier at all?

To answer this question, ask yourself what happens when we shift to a new interpretation with fewer or more things in its domain. In the new interpretation, what conjunction will have the same truth value as a given universally quantified sentence? If the new interpretation has a larger domain, our conjunction will have more conjuncts. If the new interpretation has a smaller domain, our conjunction will have fewer conjuncts. In other words, when we are looking for a conjunction of instances to give us the truth value of a universally quantified sentence, the conjunction will change from interpretation to interpretation. You can see in this way that the universal quantifier really does add something new. It acts rather like a variable conjunction sign. It has the effect of forming a long conjunction, with one conjunct for each of the objects in an interpretation's domain. If an interpretation's domain has infinitely many objects, a universally quantified sentence has the effect of an infinitely long conjunction!

What about existentially quantified sentences? All the work is really done. We repeat everything we said for universal quantification, replacing the word 'all' with 'some':



(Incomplete Definition) For any existentially quantified sentence (∃)(. . . **u**. . .), the *Substitution Instance* of the sentence, with the name **s** substituted for the variable **u** is (. . . **s** . . .), the sentence formed by dropping the initial existential quantifier and writing **s** wherever **u** had occurred.

(Incomplete Definition) An *existentially quantified sentence is true in an interpretation* just in case **some** (i.e., one or more) of the sentence's substitution instances, formed with names in the interpretation, are true in the interpretation.

As with the parallel definitions for universally quantified sentences, these definitions will have to be refined when we get to chapter 3.

To illustrate, let's see whether the sentence '(∃x)(Bx & Lxe)' is true in the interpretation of figure 2–1. We will need the sentence's substitution instances. We drop the quantifier and write in 'a' wherever we see 'x', giving 'Ba & Lae', the instance with 'a' substituted for 'x'. In the same way, we form the instance with 'e' substituted for 'x', namely, 'Be & Lee'. '(∃x)(Bx &Lxe)' is true in the interpretation just in case one or more of its substitution instances are true in the interpretation. Because 'Ba' and 'Lae' are true in the interpretation, the first instance, 'Ba & Lae', is true, and so '(∃x)(Bx & Lxe)' is true.

Have you noticed that, just as we have a connection between universal quantification and conjunction, we have the same connection between existential quantification and **disjunction:** '(∃x)(Bx & Lxe)' is true in our interpretation just in case one or more of its instances are true. But one or more of its instances are true just in case their disjunction

(Ba & Lae) v (Be & Lee)

is true. In a longer or shorter interpretation we will have the same thing with a longer or shorter disjunction. Ask yourself, when is an existentially quantified sentence true in an interpretation? It is true just in case the disjunction of all its substitution instances in that interpretation is true in the interpretation. Just as the universal quantifier acted like a variable conjunction sign, the existential quantifier acts like a variable disjunction sign. In an interpretation with an infinite domain, an existentially quantified sentence even has the effect of an infinite disjunction.

I hope that by now you have a pretty good idea of how to determine whether a quantified sentence is true or false in an interpretation. In understanding this you also come to understand everything there is to know about the meaning of the quantifiers. Remember that we explained the meaning of the sentence logic connectives '~', '&', 'v', '⊃', and '≡' by giving their truth table definitions. For example, explaining how to determine whether or not a conjunction is true in a line of a truth table tells you everything there is to know about the meaning of '&'. In the same way, our characterization of truth of a quantified sentence in an interpretation does the same kind of work in explaining the meaning of the quantifiers.

This point about the meaning of the quantifiers illustrates a more general fact. By a "case" in sentence logic we mean a line of a truth table, that is, an assignment of truth values to sentence letters. The interpretations of predicate logic generalize this idea of a case. Keep in mind that interpretations do the same kind of work in predicate logic that assignments of truth values to sentence letters do in sentence logic, and you will easily extend what you already know to understand validity, logical truth, contradictions, and other concepts in predicate logic.

By now you have also seen how to determine the truth value which an interpretation gives to any sentence, not just to quantified sentences. An interpretation itself tells you which atomic sentences are true and which are false. You can then use the rules of valuation for sentence logic connectives together with our two new rules for the truth of universally and existentially quantified sentences to determine the truth of any compound sentence in terms of the truth of shorter sentences. Multiple quantification still calls for some refinements, but in outline you have the basic ideas.

### *EXERCISES*

2–2. Consider the interpretation

D = {a,b}; ~Ba & Bb & Laa & ~Lab & Lba & ~Lbb.

For each of the following sentences, give all of the sentence's substitution instances in this interpretation, and for each substitution instance say whether the instance is true or false in the interpretation. For example, for the sentence '(∀x)Bx', your answer should look like this:

| GIVEN SENTENCE | SUBSTITUTION INSTANCES |
|---|---|
| (∀x)Bx | Ba, false in the interpretation |
| | Bb, true |

a) (∃x)Bx b) (∃x)~Lxa c) (∀x)Lxa

d) (∃x)Lbx e) (∀)(Bx v Lax) f) (∃x)(Lxa & Lbx)

g) (∀x)(Bx ⊃ Lbx) h) (∃x)[(Lbx & Bb) v Bx]

i) (∀x)[Bx ⊃ (Lxx ⊃ Lxa)]



j) (∀x)[(Bx v Lax) ⊃ (Lxb v ~Bx)]

k) (∃x)[(Lax & Lxa) ≡ (Bx v Lxb)]

2–3. For each of the sentences in exercise 2–2, say whether the sentence is true or false in the interpretation of exercise 2–2.

2–4. For each of the following sentences, determine whether the sentence is true or false in the interpretation of exercise 2–2. In this exercise, you must carefully determine the main connective of a sentence before applying the rules to determine its truth in an interpretation. Remember that a quantifier is a connective which applies to the **shortest** full sentence which follows it. Remember that the main connective of a sentence is the **last** connective that gets used in building the sentence up from its parts. To determine whether a sentence is true in an interpretation, first determine the sentence's main connective. If the connective is '&', 'v', '~', '⊃', or '≡', you must first determine the truth value of the components, and then apply the rules for the main connective (a conjunction is true just in case both conjuncts are true, and so on). If the main connective is a quantifier, you have to determine the truth value of the substitution instances and then apply the rule for the quantifier, just as you did in the last exercise.

a) (∃x)Lxx ⊃ (∀x)(Bx v Lbx)
b) ~(∃x)(Lxx ⊃ Bx) & (∀x)(Bx ⊃ Lxx)
c) (∃x)[Bx ≡ (Lax v Lxb)]
d) (∃x)(Lxb v Bx) ⊃ (Lab v ~Ba)
e) ~(∀x)(~Lxx v Lxb) ⊃ (Lab v ~Lba)
f) (∃x)[(Lbx v Bx) ⊃ (Lxb & ~Bx)]
g) (∀x)~[(~Lxx ≡ Bx) ⊃ (Lax ≡ Lxa)]
h) (∀x)(Lax v Lxb) v (∃x)(Lax v Lxb)
i) (∃x)[Lxx & (Bx ⊃ Laa)] & (∃x)~(Lab ≡ Lxx)
j) (∀x){[Bx v (Lax & ~Lxb)] ⊃ (Bx ⊃ Lxx)}

2–5. In the past exercises we have given interpretations by explicitly listing objects in the domain and explicitly saying which predicates apply to which things. We can also describe an interpretation in more general terms. For example, consider the interpretation given by

i) Domain: All U.S. citizens over the age of 21.
ii) Names: Each person in the domain is named by 'a' subscripted by his or her social security number.
iii) Predicates: Mx: x is a millionaire.
Hx: x is happy.

(That is, a one place predicate 'Mx' which holds of someone just in case that person is a millionaire and a one place predicate 'Hx' which holds of someone just in case that person is happy.)

a) Determine the truth value of the following sentences in this interpretation. In each case explain your answer. Since you can't write out all the substitution instances, you will have to give an informal general statement to explain your answer, using general facts you know about being a millionaire, being happy, and the connection (or lack of connection) between these.

a1) $(\exists x)Mx$ a2) $(\forall x)Hx$ a3) $(\forall x)(Hx \supset Mx)$ a4) $(\exists x)(Mx \& \sim Hx)$

a5) $(\forall x)[(Mx \supset Hx) \& (\sim Mx \supset \sim Hx)]$

a6) $(\exists x)[(Hx \& Mx) \vee (\sim Hx \& \sim Mx)]$

a7) $(\exists x)(Mx \& Hx) \& (\exists x)(Mx \& \sim Hx)$

a8) $(\forall x)(Hx \supset Mx) \supset \sim (\exists x)Mx$

Here is another example:

i) Domain: All integers, 1, 2, 3, 4, . . .
ii) Names: Each integer is named by 'a' subscripted by that integer's numeral. For example, 17 is named by '$a_{17}$'.
iii) Predicates: Ox: x is odd.
Kxy: x is equal to or greater than y.

b) Again, figure out the truth value of the following sentences, and explain informally how you arrived at your answer.

b1) $(\exists x)Ox$ b2) $(\forall x)\sim Ox$ b3) $(\exists x)(Ox \& Kxx)$

b4) $(\forall x)Kxa_{17}$ b5) $(\forall x)(Ox \vee \sim Ox)$

b6) $(\exists x)(Ox \& Kxa_{17})$

b7) $(\forall x)[Ox \equiv (\sim Kxa_{18} \& Kxa_{17})]$

b8) $(\exists x)(Kxa_{17} \supset Kxa_{18}) \& (\forall x)(\sim Kxa_{17} \vee Kxa_{18})$

b9) $(\forall x)(Ox \supset Kxa_{17}) \& (\forall x)(\sim Ox \supset \sim Kxa_{17})$

## 2-3. VALIDITY IN PREDICATE LOGIC

In sentence logic, we said that an argument is valid if and only if, for all possible cases in which all the premises are true, the conclusion is true also. In predicate logic, the intuitive notion of validity remains the same. We change things only by generalizing the notion of possible case. Where before we meant that all lines in the truth table which made all premises



true also make the conclusion true, now we mean that all interpretations which make all the premises true also make the conclusion true:

> An argument expressed with sentences in predicate logic is valid if and only if the conclusion is true in every interpretation in which all the premises are true.

You may remember that we got started on predicate logic at the beginning of chapter 1 because we had two arguments which seemed valid but which sentence logic characterized as invalid. To test whether predicate logic is doing the job it is supposed to do, let us see whether predicate logic gives us the right answer for these arguments;

| | |
|---|---|
| Everyone loves Eve. | $(\forall x)Lxe$ |
| Adam loves Eve. | Lae |

Suppose we have an interpretation in which '$(\forall x)Lxe$' is true. Will 'Lae' have to be true in this interpretation also? Notice that 'Lae' is a substitution instance of '$(\forall x)Lxe$'. A universally quantified sentence is true in an interpretation just in case all its substitution instances are true in the interpretation. So in any interpretation in which '$(\forall x)Lxe$' is true, the instance 'Lae' will be true also. And this is just what we mean by the argument being valid.

Let's examine the other argument:

| | |
|---|---|
| Adam loves Eve. | Lae |
| Someone loves Eve. | $(\exists x)Lxe$ |

Suppose we have an interpretation in which 'Lae' is true. Does '$(\exists x)Lxe$' have to be true in this interpretation? Notice that 'Lae' is an instance of '$(\exists x)Lxe$'. We know that '$(\exists x)Lxe$' is true in an interpretation if even one of its instances is true in the interpretation. Thus, if 'Lae' is true in an interpretation, '$(\exists x)Lxe$' will also be true in that interpretation. Once again, the argument is valid.

Along with validity, all our ideas about counterexamples carry over from sentence logic. When we talked about the validity of a sentence logic argument, we first defined it in this way: An argument is valid just in case any line of the truth table which makes all the premises true makes the conclusion true also. Then we reexpressed this by saying: An argument is valid just in case it has no counterexamples; that is, no lines of the truth table make all the premises true and the conclusion false. For predicate logic, all the ideas are the same. The only thing that has changed is that we now talk about interpretations where before we talked about lines of the truth table:

A *Counterexample* to a predicate logic argument is an interpretation in which the premises are all true and the conclusion is false.

A predicate logic argument is *Valid* if and only if it has no counterexamples.

Let's illustrate the idea of counterexamples in examining the validity of

$$\frac{\text{Lae}}{(\exists x)\text{Lxe}}$$

Is there a counterexample to this argument? A counterexample would be an interpretation with 'Lae' true and '(∃x)Lxe' false. But there can be no such interpretation. 'Lae' is an instance of '(∃x)Lxe', and '(∃x)Lxe' is true in an interpretation if even one of its instances is true in the interpretation. Thus, if 'Lae' is true in an interpretation, '(∃x)Lxe' will also be true in that interpretation. In other words, there can be no interpretation in which 'Lae' is true and '(∃x)Lxe' is false, which is to say that the argument has no counterexamples. And that is just another way of saying that the argument is valid.

For comparison, contrast the last case with the argument

$$\frac{(\exists x)\text{Bx}}{\text{Ba}}$$

It's easy to construct a counterexample to this argument. Any case in which someone other than Adam is blond and Adam is not blond will do the trick. So an interpretation with Adam and Eve in the domain and in which Eve is blond and Adam is not blond gives us a counterexample, showing the argument to be invalid.

This chapter has been hard work. But your sweat will be repaid. The concepts of interpretation, substitution instance, and truth in an interpretation provide the essential concepts you need to understand quantification. In particular, once you understand these concepts, you will find proof techniques for predicate logic to be relatively easy.

### EXERCISES

2–6. For each of the following arguments, determine whether the argument is valid or invalid. If it is invalid, show this by giving a counterexample. If it is valid, explain your reasoning which shows it to be valid. Use the kind of informal reasoning which I used in discussing the arguments in this section.



You may find it hard to do these problems because I haven't given you any very specific strategies for figuring out whether an argument is valid. But don't give up! If you can't do one argument, try another first. Try to think of some specific, simple interpretation of the sentences in an argument, and ask yourself—"Are the premise and conclusion both true in that interpretation?" Can I change the interpretation so as to make the premise true and the conclusion false? If you succeed in doing that, you will have worked the problem because you will have constructed a counterexample and shown the argument to be invalid. If you can't seem to be able to construct a counterexample, try to understand why you can't. If you can see why you can't and put this into words, you will have succeeded in showing that the argument is valid. Even if you might not succeed in working many of these problems, playing around in this way with interpretations, truth in interpretations, and counterexamples will strengthen your grasp of these concepts and make the next chapter easier.

a) $\dfrac{(\forall x)\text{Lxe}}{(\exists x)\text{Lxe}}$ b) $\dfrac{\text{Lae}}{(\forall x)\text{Lxe}}$ c) $\dfrac{(\exists x)\text{Lxe}}{\text{Lae}}$

d) $\dfrac{(\forall x)(\text{Bx} \,\&\, \text{Lxe})}{(\forall x)\text{Bx}}$ e) $\dfrac{(\forall x)(\text{Bx} \supset \text{Lxe})}{(\exists x)\text{Bx}}$

f) $\dfrac{(\exists x)\text{Bx} \,\&\, (\exists x)\text{Lxa}}{(\exists x)(\text{Bx} \,\&\, \text{Lxa})}$ g) $\dfrac{(\forall x)(\text{Bx} \supset \text{Lxe}) \,\&\, (\forall x)(\sim\text{Bx} \supset \text{Lxa})}{(\forall x)[(\text{Bx} \supset \text{Lxe}) \,\&\, (\sim\text{Bx} \supset \text{Lxa})]}$

### CHAPTER SUMMARY EXERCISES

Provide short explanations for each of the following, checking against the text to make sure you understand each term clearly and saving your answers in your notebook for reference and review.

a) Interpretation
b) Interpretation of a Sentence
c) Substitution Instance
d) Truth in an Interpretation
e) Validity of a Predicate Logic Argument