# Lecture 2

**Example** $\{\}$ is the empty set.

To avoid any confusion, we give this set a special name $\emptyset = \{\}$.

**Example** $\{\} \neq \{\emptyset\}$

empty set contains no elements

set containing the emptyset contains 1 element

**Definition** Let $A$ be a set.

The cardinality of $A$, written $|A|$, is the number of elements in $A$.

**Examples**

(1) $|\emptyset| = 0$.

(2) $|\{1,2,3\}| = 3$.

Remark

A lot of probability theory involves calculating $|A|$ for special sets $A$.

Examples Some special sets we shall need in this course.

(1) $\mathbb{N} = \{0, 1, 2, \underbrace{\ldots}_{\text{"and so on"}}\}$ the natural numbers.

(2) $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

the integers (Zahlen - German for integers)

(3) $\mathbb{Q}$ all positive and negative fractions or quotients. The rational numbers.

(4) $\mathbb{R}$ all real numbers. These can be written

decimal point in UK

$$a \cdot \underbrace{a_1 a_2 a_3 \ldots}_{\text{potentially infinitely many digits after the decimal point}}$$

integer

(5) Later on in this course, we shall study the <u>Complex numbers</u> $\mathbb{C}$.

A very common way to define a set is to describe a property that the elements of the set are to have.

<u>Example</u>

such that

$$\mathbb{P} = \{a : a \text{ is a prime}\}$$

is the set of prime numbers.

<u>Examples</u>

(1) $A = \{a : a \in \mathbb{R} \text{ and } a^2 = 1\}$

$= \{-1, 1\}$.

(2) $B = \{a : a \in \mathbb{N} \text{ and } a^2 = 1\}$

$= \{1\}$.

(3) $C = \{a : a \in \mathbb{Q} \text{ and } a^2 = 2\}$

$= \emptyset$

needs a proof.

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Given a set $A$ we can form sets $B$ whose elements are chosen from those of $A$. We say that $B$ is a <u>subset</u> of $A$ written

$B \subseteq A$.

<u>Examples</u>

Mathematicians use the word 'choose' in a special way.

(1) $\emptyset \subseteq A$. Choose no elements from $A$.

(2) $A \subseteq A$. Choose all elements from $A$!

**Definition** Let $A$ be a set. $P(A) = \{B : B \subseteq A\}$ is called the **powerset** of $A$. It is the set of all subsets of $A$.

**Example** Let $A = \{1, 2, 3\}$. Find $P(A)$.

| Cardinality of subset | subsets |
|---|---|
| 0 | $\emptyset$ |
| 1 | $\{1\}, \{2\}, \{3\}$ |
| 2 | $\{2,3\}, \{1,3\}, \{1,2\}$ |
| 3 | $\{1,2,3\}$ |

$\therefore P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{2,3\}, \{1,3\}, \{1,2\}, \{1,2,3\}\}$.

Observe that

$$\left| P(\{1,2,3\}) \right| = 8.$$

There is an interesting number pattern here.

| Cardinality of subset | number of such subsets |
|---|---|
| 0 | 1 |
| 1 | 3 |
| 2 | 3 |
| 3 | 1 |

So, $8 = 1 + 3 + 3 + 1$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
. . . .

Pascal's $\triangle$

This is not a Coincidence

And, also.

$$(a+b)^3 = (a+b)(a+b)^2$$

$$= (a+b)(a^2 + 2ab + b^2)$$

$$= a^3 + 2a^2b + ab^2 + ba^2 + 2ab^2 + b^3$$

$$= \underline{\underline{1}}\,a^3 + \underline{\underline{3}}\,a^2b + \underline{\underline{3}}\,ab^2 + \underline{\underline{1}}\,b^3$$

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$\boxed{\text{Why does the powerset "know" about Pascal's } \triangle \text{?}}$