Homework 1: other solutions

Question 1

- S'pose that $\sqrt{5}$ is rational. Then we can write $\sqrt{5} = \frac{a}{b}$ where $a, b \in \mathbb{N}$. Squaring both sides and reorganizing we get $5b^2 = a^2$.
Clearly, $b \neq 1$. So, by the fundamental theorem of arithmetic (FTA) we can write

$$b = p_1^{n_1} \cdots p_r^{n_r}$$

uniquely as a product of primes. Thus

$$b^2 = p_1^{2n_1} \cdots p_r^{2n_r}.$$

We therefore have that

$$a^2 = 5p_1^{2n_1} \cdots p_r^{2n_r}.$$

Clearly, $a \neq 1$. So, by the (FTA) we can write

$$a = q_1^{m_1} \cdots q_s^{m_s}$$

uniquely as a product of primes. Thus

$$a^2 = q_1^{2m_1} \cdots q_s^{2m_s}.$$

We therefore have that

$$\underbrace{q_1^{2m_1} \cdots q_s^{2m_s}}_{\text{even no. of primes}} = \underbrace{5p_1^{2n_1} \cdots p_r^{2n_r}}_{\text{odd no. of primes}}.$$

This contradicts (FTA). It follows that $\sqrt{5}$ is irrational. ▨

- S'pose that $\sqrt{5}$ is rational. Then we can write $\sqrt{5} = \frac{a}{b}$, where a and b have no common divisors.

As before, $5b^2 = a^2$.

If b^* is even then a^2 is even and so a is even (using previous results). It follows that b must be odd. S'pose that a were even. Put $b = 2m+1$.

Then $5(4m^2 + 4m+1) = a^2$ (even).

It follows that $\underbrace{(20m^2 + 20m + 4) + 1}_{\text{odd}} = a^2$.

Thus a must be odd.

We have shown that both a and b are odd.

Let $a = 2n+1$ and $b = 2m+1$.

Then $5(2m+1)^2 = (2n+1)^2$.

Expanding $5(4m^2 + 4m+1) = 4n^2 + 4n+1$

So, $20m^2 + 20m + 4 + 1 = 4n^2 + 4n + 1$.

This gives $20m^2 + 20m + 4 = 4n^2 + 4n$.

Dividing by 4 we get

$$\underbrace{5m^2 + 5m + 1}_{\text{odd}} = n^2 + n = \underbrace{n(n+1)}_{\text{even}}.$$

This is a contradiction and so $\sqrt{5}$ is irrational. $\blacksquare$

Question 2

- To show that the last digit of n^5 = the last digit of n, it is equivalent to show that $10 \mid n^5 - n$. Two ways of doing this.

By induction statement true when $n=0$.

Assume $10 \mid k^5 - k$. We prove that $10 \mid (k+1)^5 - (k+1)$. By the binomial theorem

$$(k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 4k.$$

$= (k^5 - k) + 5k^4 + 10k^3 + 10k^2 + 5k$

Clearly, $5 \mid (k+1)^5 - (k+1)$.

To see that $2 \mid (k+1)^5 - (k+1)$

Observe that $2 \mid (k^5 - k)$ by (IH)

and $2 \mid (5k^4 + 10k^3 + 10k^2 + 5k)$

since if k is odd this is a sum of four odd nos. ∎

12) By expansion

$$n^5 - n = (n-1)\,n\,(n+1)\,(n^2+1)$$

Clearly, $2 \mid n^5 - n$.

To see that $5 \mid (n-1)\,n\,(n+1)\,(n^2+1)$

observe that if $5 \nmid (n-1)$, $5 \nmid n$, $5 \nmid (n+1)$.

then $n = 5q+2$ or $n = 5q+3$.

Then $n^2+1 = 25q^2 + 10q + 5$

or

$n^2+1 = 25q^2 + 10q + 10$

In both cases, $5 \mid n^2+1$. ■