Properties of matrices

This is a version of part of Section 8.2.

Properties of matrix addition

We restrict attention to the set of all $m \times n$ matrices.

(MA1): $(A + B) + C = A + (B + C)$. This is the *associative law* for matrix addition.

(MA2): $A + O = A = O + A$. The zero matrix O, the same size as A, is the *additive identity* for matrices the same size as A.

(MA3): $A + (-A) = O = (-A) + A$. The matrix $-A$ is the unique *additive inverse* of A.

(MA4): $A + B = B + A$. Matrix addition is *commutative*.

Thus matrix addition has the same properties as the addition of real numbers, apart from the fact that the sum of two matrices is only defined when they have the same size.

Properties of matrix multiplication

(MM1): The product $(AB)C$ is defined precisely when the product $A(BC)$ is defined, and when they are both defined $(AB)C = A(BC)$. This is the *associative law* for matrix multiplication.

(MM2): Let A be an $m \times n$ matrix. Then $I_m A = A = A I_n$. The matrices I_m and I_n are the *left and right multiplicative identities*, respectively. It is important to observe that for matrices that are not square different identities are needed on the left and on the right.

(MM3): $A(B + C) = AB + AC$ and $(B + C)A = BA + CA$ when the products and sums are defined. These are the *left and right distributivity laws*, respectively, for matrix multiplication over matrix addition.

Thus, apart from the fact that it is not always defined, matrix multiplication has the same properties as the multiplication of real numbers **except for** the following three major differences.

(1) *Matrix multiplication is not commutative.* For example, if

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

then $AB \neq BA$.

(2) *The product of two matrices can be a zero matrix without either matrix being a zero matrix.* For example, if

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} -2 & -6 \\ 1 & 3 \end{pmatrix}$$

then $AB = O$.

(3) *Cancellation of matrices is not allowed in general.* For example, if

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \text{ and } C = \begin{pmatrix} -1 & 1 \\ 1 & 4 \end{pmatrix}$$

then $A \neq O$ and $AB = AC$ but $B \neq C$.

Scalar multiplication causes us no problems.

Properties of scalar multiplication

(S1): $1A = A$ and $-1A = -A$.
(S2): $0A = O$.
(S3): $\lambda(A + B) = \lambda A + \lambda B$ where λ is a scalar.
(S4): $(\lambda\mu)A = \lambda(\mu A)$ where λ and μ are scalars.
(S5): $(\lambda + \mu)A = \lambda A + \mu A$ where λ and μ are scalars.
(S6): $(\lambda A)B = A(\lambda B) = \lambda(AB)$ where λ is a scalar.

The transpose is also straightforward apart from property (T4) below.

Properties of the transpose

(T1): $(A^T)^T = A$.
(T2): $(A + B)^T = A^T + B^T$.
(T3): $(\lambda A)^T = \lambda A^T$ where λ is a scalar.
(T4): $(AB)^T = B^T A^T$.

There are some important consequences of the above properties.

- Because matrix addition is associative, we can apply generalized associativity and write sums without brackets. Similarly, because matrix multiplication is associative, we can apply generalized associativity and write matrix products without brackets, though ensuring that we keep the same order.
- The left and right distributivity laws can be extended to arbitrary finite sums.