Solutions 1

(1) (a) C was a knight. The argument runs as follows. If you ask a knight what he is he will say he is a knight. If you ask a knave what he is he is obliged to lie and so also say that he is a knight. Thus no one on this island can say they are a knave. This means that B is a knave. Hence C was correct in saying that B lies and so C was a knight.

(b) A is a knave, B a knight and C a knave. The argument runs as follows. If all three were knaves then C would be telling the truth which contradicts the fact that he is a knave. Thus at least one of the three is a knight and at most two are knights. It follows that C is a knave. Suppose that there were exactly two knights. Then both A and B would be knights but they contradict each other. It follows that exactly one of them is a knight. Hence A is knave and B is a knight.

(2) Sam drinks water and Mary owns the aardvark. The following table shows all the information you should have deduced.

	1	2	3	4	5
House	Yellow	Blue	Red	White	Green
Pet	Fox	Horse	Snails	Dog	Aardvark
Name	Sam	Tina	Sarah	Charles	Mary
Drink	Water	Tea	Milk	Orange juice	Coffee
Car	Bentley	Chevy	Oldsmobile	Lotus	Porsche

The starting point is the following table.

	1	2	3	4	5
House					
Pet					
Name					
Drink					
Car					

Using clues (h), (i) and (n), we can make the following entries in the table.

	1	2	3	4	5
House		Blue			
Pet					
Name	Sam				
Drink			Milk		
Car					

There are a number of different routes from here. I shall just give some examples of how you can reason. Clue (a) tells us that Sarah lives in the red house. Now she cannot live in the first house, because Sam lives there, and she cannot live in the second house because that is blue. We are therefore left with the following which summarizes all the possibilities so far.

	1	2	3	4	5
House	Yellow? White? Green?	Blue	Red?	Red?	Red?
Pet					
Name	Sam		Sarah?	Sarah?	Sarah?
Drink			Milk		
Car					

Clue (b) tells us that Charles owns the dog. It follows that Sam cannot own the dog. We are therefore left with the following possibilities.

	1	2	3	4	5
House	Yellow? White? Green?	Blue	Red?	Red?	Red?
Pet	Fox? Horse? Snails? Aardvark?				
Name	Sam		Sarah?	Sarah?	Sarah?
Drink			Milk		
Car					

(3) There are two possible secret numbers consistent with the information given: 2745 or 4725.

(4) This is called the *Collatz problem* or the $3x+1$ *problem*. Nobody has yet found a proof. It is therefore conceivable that there is a number where the process described in the question does not terminate. I included this question to show that unsolved problems are not limited to what you might regard as advanced mathematics.

(5) This difficult question is taken from the famous book: *Gödel, Escher, Bach*. The answer is no. The key is to focus on the number of Is in a string which we call the I*-count*. Rule-I does

not change the I-count. Rule-II doubles the I-count. Rule-III reduces the I-count by 3. Rule-IV does not change the I-count. We begin with a string whose I-count is 1 and our goal is to obtain a string whose I-count is 0. The problem reduces to showing that applying the above rules to a string whose I-count is 1 never results in a string whose I-count is 0. This amounts to showing that if 3 does not divide n then 3 does not divide $2n$, and if 3 does not divide n then 3 does not divide $n - 3$.

(6) Here is the completed puzzle. It was taken from *Solving sudoku* by Michael Mepham available at
`www.sudoku.org.uk/PDF/Solving_Sudoku.pdf`.

8	1	3	4	2	9	7	6	5
4	6	2	5	7	1	8	3	9
7	9	5	3	6	8	1	4	2
2	4	7	1	5	3	9	8	6
5	3	9	8	4	6	2	1	7
6	8	1	2	9	7	4	5	3
9	7	8	6	1	5	3	2	4
1	2	6	7	3	4	5	9	8
3	5	4	9	8	2	6	7	1

(7) Draw a diagonal across the quadrilateral dividing the figure into two triangles. The sum of the interior angles of the figure is equal to the sum of the angles in the two triangles. This is $360°$.

(8) (a) Two applications of Pythagoras' theorem give

$$\sqrt{2^2 + 3^2 + 7^2} = \sqrt{62}.$$

(b) Draw a diagonal of the square and then construct the square with side that diagonal. This has twice the area by Pythagoras' theorem.

(c) The area of such a triangle is $\frac{1}{2}xy$ but we are told that it equals $\frac{1}{4}z^2$. Hence $\frac{1}{2}xy = \frac{1}{4}z^2$. By Pythagoras' theorem $z^2 = x^2 + y^2$. Substituting this value for z^2 we get that $\frac{1}{2}xy = \frac{1}{4}(x^2 + y^2)$. Rearrange this to get $(x - y)^2 = 0$. Hence $x - y = 0$ and so $x = y$. It follows that the triangle is isosceles.

(9) Draw up a table with three columns and a number of rows. The first row is labelled 0,1,2. the second is labelled 3,4,5 and so forth. Circle those numbers that can be written in the form $3x + 5y$ where $x, y \in \mathbb{N}$. We say that a row is complete if all numbers in that row are circled. Observe that $9 = 3 \cdot 3$ and

$10 = 2 \cdot 5$ and $11 = 2 \cdot 3 + 5$. Since this row is complete all subsequent rows are complete. Now $8 = 3 + 5$ but 7 cannot be written in the given way. It follows that 7 is the largest value that cannot be made.