# Solutions 10

(1) (a) $BD = \mathbf{c} - \mathbf{a}$.
(b) $AE = \mathbf{a} + \mathbf{c}$.
(c) $DE = \mathbf{a}$.
(d) $CF = \mathbf{c}$.
(e) $AC = \mathbf{a} + \mathbf{b}$.
(f) $BF = \mathbf{b} + \mathbf{c}$.

(2) If the quadrilateral is a parallelogram then $\mathbf{a} = -\mathbf{c}$. Conversely, suppose that $\mathbf{a} + \mathbf{c} = \mathbf{0}$. Then because $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = \mathbf{0}$ we deduce that $\mathbf{b} + \mathbf{d} = \mathbf{0}$ and so the shape is a parallelogram.

(3) (a) $EA = -(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d})$.
(b) $DA = -(\mathbf{a} + \mathbf{b} + \mathbf{c})$.
(c) $DB = -(\mathbf{b} + \mathbf{c})$.
(d) $CA = -(\mathbf{a} + \mathbf{b})$.
(e) $EC = -(\mathbf{c} + \mathbf{d})$.
(f) $BE = \mathbf{b} + \mathbf{c} + \mathbf{d}$.

(4) The remaining sides are: $\mathbf{b} - \mathbf{a}$, $-\mathbf{a}$, $-\mathbf{b}$, $\mathbf{a} - \mathbf{b}$. This was obtained using the given subdivision of the hexagon into equilateral triangles and then observing which lines were parallel to each other.

(5) Calculate

$$(\|\mathbf{a}\| \mathbf{b} + \|\mathbf{b}\| \mathbf{a}) \cdot (\|\mathbf{a}\| \mathbf{b} - \|\mathbf{b}\| \mathbf{a})$$

using distributivity and the fact that $\|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{a}$. The answer is zero, and so the vectors are orthogonal.

(6) Calculate

$$\left(\mathbf{b} - \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \mathbf{a}\right) \cdot \mathbf{a}$$

using the distributive law, the fact that $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$, and the fact that $\|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{a}$ to get 0.

(7) (a) $\mathbf{0}$.
(b) First we expand using distributivity

$$(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v}) = \mathbf{u} \times \mathbf{u} + \mathbf{v} \times \mathbf{u} - \mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{v}.$$

But $\mathbf{u} \times \mathbf{u} = \mathbf{0} = \mathbf{v} \times \mathbf{v}$ and $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$. Thus the answer is $2\mathbf{v} \times \mathbf{u}$.

(8) We calculate

$$\mathbf{a} \cdot (2\mathbf{b} - \mathbf{a}) = 2\mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{a}$$

but $\mathbf{a} \cdot \mathbf{a} = 1$ and $\mathbf{a} \cdot \mathbf{b} = \cos \frac{\pi}{3} = \frac{1}{2}$. The result now follows.

(9) The lefthand side is $(\mathbf{u} - \mathbf{v})^2 + (\mathbf{u} + \mathbf{v})^2$ which expands to

$$\mathbf{u}^2 - 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v}^2 + \mathbf{u}^2 + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v}^2$$

which is equal to

$$2(\mathbf{u}^2 + \mathbf{v}^2)$$

as required. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors lying along two adjacent sides of the parallelogram. The diagonals are $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$. The result now follows.

(10) (a) $\sqrt{6}$ and $\sqrt{14}$.
(b) $3\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$.
(c) $\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.
(d) 7.
(e) $40°$.
(f) $\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$.
(g) $\frac{1}{\sqrt{35}}(\mathbf{i} - 5\mathbf{j} + 3\mathbf{k})$.

(11) $(\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}$, whereas $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = -\mathbf{k}$. Thus the vector product is not associative.

(12) 0 in both cases. This is a useful check when calculating vector products.

(13) A diagonal is $\mathbf{i} + \mathbf{j} + \mathbf{k}$. The cosine of the angle between this vector and $\mathbf{i}$, one of the edges, is $\frac{1}{\sqrt{3}}$. Thus the angle is between 54 and 55 degrees.

(14) The number $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is the determinant

$$\begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix}$$

which is equal to 49.