Solutions 11

(1) (a) We begin by finding the parametric equation of the line through the two given points. Let $\mathbf{r}$ be the position vector of a point on the line. Then the vectors

$$\mathbf{r} - (\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$

and

$$(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) - (\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$

are parallel. Thus there is a scalar λ such that

$$\mathbf{r} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}).$$

This is the parametric equation of the line.
To obtain the non-parametric equation, we first equate components in the above equation and get

$$x = 1 + \lambda, \quad y = -1 + 4\lambda, \quad z = 2 + 2\lambda.$$

Now we eliminate the parameter λ to get the non-parametric equations

$$x - 1 = \frac{y+1}{4}, \quad \frac{y+1}{4} = \frac{z-2}{2}.$$

(b) We begin by finding the parametric equation of the plane through the three given points. First we must find two vectors that are parallel to the plane. Let

$$\mathbf{a} = (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) - (\mathbf{i} + 3\mathbf{k}) = 2\mathbf{j} - 4\mathbf{k}$$

and

$$\mathbf{b} = (3\mathbf{i} - \mathbf{j} - 2\mathbf{k}) - (\mathbf{i} + 3\mathbf{k}) = 2\mathbf{i} - \mathbf{j} - 5\mathbf{k}.$$

Thus if $\mathbf{r}$ is the position vector of a point in the given plane then we have that

$$\mathbf{r} - (\mathbf{i} + 3\mathbf{k}) = \lambda\mathbf{a} + \mu\mathbf{b}$$

for some parameters λ and μ. Hence the parametric equation of the plane is

$$\mathbf{r} = \mathbf{i} + 3\mathbf{k} + \lambda(2\mathbf{j} - 4\mathbf{k}) + \mu(2\mathbf{i} - \mathbf{j} - 5\mathbf{k}).$$

To find the non-parametric equation of the given plane, we need to find a vector normal to the plane. The vector $\mathbf{a} \times \mathbf{b}$ will do the trick. This is equal to $-14\mathbf{i} - 8\mathbf{j} - 4\mathbf{k}$. Thus if $\mathbf{r}$ is the position vector of a point in the given plane we have that

$$(\mathbf{r} - (\mathbf{i} + 3\mathbf{k})) \cdot (14\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}) = 0.$$

This simplifies to

$$7x + 4y + 2z = 13.$$

(c) $x + y - z = 3$.

(2) (a) $\mathbf{r} = 4\mathbf{i} + 5\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})$.

(b) $\mathbf{r} = 5\mathbf{i} - 4\mathbf{j} + \mu(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$.

(c) They do intersect and the position vector of the point of intersection is $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$.

(3) $\frac{1}{5}(7\mathbf{i} - 8\mathbf{j}) + \frac{\lambda}{5}(2\mathbf{i} - 8\mathbf{j} + 5\mathbf{k})$. But this could equally well be written $\frac{1}{5}(7\mathbf{i} - 8\mathbf{j}) + \mu(2\mathbf{i} - 8\mathbf{j} + 5\mathbf{k})$ where $\mu \in \mathbb{R}$.

(4) Let θ be the angle that $\mathbf{q} - \mathbf{p}$ makes with the direction determined by $\mathbf{d}$. Then the required distance is $||\mathbf{q} - \mathbf{p}|| \sin\theta$. This quickly leads to the result.

(5) Let θ be the angle the vector $\mathbf{q} - \mathbf{p}$ makes with the normal $\mathbf{n}$. Then the required distance is $||\mathbf{q} - \mathbf{p}|| \cos\theta$. This quickly leads to the result.

(6) (a) The desired equation is $(x-1)^2 + (y-1)^2 + (z-1)^2 = 4$ (which can be multiplied out).

(b) Write first in the form $(x-1)^2 - 1 + (y-2)^2 - 4 + (z-3)^2 - 9 - 2 = 0$ by completing the square. This gives us $(x-1)^2 + (y-2)^2 + (z-3)^2 = 16$. Thus the centre is $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and the radius is 4.