# Solutions 5

(1) (a) Let $A$ be the set of starters, $B$ the set of main courses, and $C$ the set of drinks. Then an element of the set $A \times B \times C$ consists of a starter, followed by a main course, followed by a drink. Thus the set $A \times B \times C$ is the set of all possible such dinners. The cardinality of this set is $2 \cdot 3 \cdot 4$. Thus there are 24 such dinners.

(b) The argument is the same as for (1) above. The number of possible dates is therefore $31 \cdot 12 \cdot 3000 = 1,116,000$.

(c) This is a question about permutations. The answer is $10!$.

(d) This is the same as the previous answer.

(e) This is a question about 3-permutations. The answer is $8 \cdot 7 \cdot 6 = 336$.

(f) This is a question about combinations. The answer is $\binom{52}{13}$.

(g) This is a question about combinations. The answer is $\binom{10}{4}$.

(h) Order matters but repetition is not allowed and so this is a question about 3-permutations. The answer is is $9 \times 8 \times 7 = 504$ ways. I should add that order is implicit in stating the rôles: chairman, secretary and treasurer.

(i) Since order is not important and repetitions are not allowed, this is a question about combinations and so the solution is

$$\binom{49}{6} = 13,983,816.$$

(j) We are just counting sequences and so the solution is $5^4 = 625$.

(2) Think of a novel as one long string of symbols. This string has length

$$250 \times 45 \times 60 = 675,000.$$

But each symbol can be one of 100 possibilities and so the number of possible novels is

$$100^{675,000}.$$

It's more convenient to write this as a power of 10 and so we get

$$10^{1,350,000}$$

possible novels. For comparison purposes, the number of atoms in the universe is estimated to be $10^{80}$.

(3) (a) This is an important result. I would be happy with a simple counting argument using Venn diagrams (the intersection gets counted twice). A more formal answer uses the fact that if $X$ and $Y$ are *disjoint* sets then $|X \cup Y| = |X| + |Y|$. Observe that

$$A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$$

where the union on the RHS is disjoint. Thus

$$|A \cup B| = |A \setminus B| + |A \cap B| + |B \setminus A|\,.$$

Now $A = (A \setminus B) \cup (A \cap B)$, which is a disjoint union, and so

$$|A| = |A \setminus B| + |A \cap B|\,.$$

Similarly

$$|B| = |B \setminus A| + |A \cap B|\,.$$

Thus

$$|A \setminus B| = |A| - |A \cap B|$$

and

$$|B \setminus A| = |B| - |A \cap B|\,.$$

Substituting this in the our first expression for $|A \cup B|$ gives the result.

(b) To prove the second claim, we calculate as follows. We use the properties of the Boolean set operations. First,

$$|(A \cup B) \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C|\,.$$

We can appeal to our result above to get

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |(A \cup B) \cap C|\,.$$

Finally, we deal the last term by writing $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. We therefore get

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|\,.$$

(4) (a)

$$\sum_{i=0}^{8} \binom{8}{i} x^i.$$

(b)

$$\sum_{i=0}^{8} \binom{8}{i} (-1)^i x^i.$$

(c) The coefficient is $\binom{10}{2}$.

(d) The coefficient is $\binom{6}{3} \cdot 3^3 \cdot 4^3$.

(e) The binomial expansion is

$$\sum_{i=0}^{9} \binom{9}{i} \left(3x^2\right)^{9-i} \left(-\frac{1}{2x}\right)^i.$$

Expanding each term carefully we get

$$\sum_{i=0}^{9} \binom{9}{i} \cdot (-1)^i \cdot 3^{9-i} \cdot 2^{-i} \cdot x^{18-2i} \cdot x^{-i}.$$

This is equal to

$$\sum_{i=0}^{9} \binom{9}{i} \cdot (-1)^i \cdot 3^{9-i} \cdot 2^{-i} \cdot x^{18-3i}.$$

We need to calculate the coefficient of $x^3$. This means we need to calculate the coefficient where $i = 5$. This is $-\binom{9}{5} \cdot 3^4 \cdot 2^{-5}$. We need to calculate the value of the constant term. This means we need to calculate the term where $i = 6$. This is $\binom{9}{6} \cdot 3^3 \cdot 2^{-6}$.

(5) (a) Put $x = y = 1$ in the binomial theorem.
(b) Put $x = 1$ and $y = -1$ in the binomial theorem.
(c) Put $x = 1$ and $y = \frac{1}{2}$ in the binomial theorem.